

Infinite Series: Ramifications

G. M. Fichtenholz

Revised English Edition
Translated and Freely Adapted by
Richard A. Silverman

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BY

G. M. FICHTENHOLZ

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Preface

The present volume of The Pocket Mathematical Library continues the study of infinite series begun in its companion volume *Infinite Series: Rudiments*, by the same author. Together the two volumes give a detailed treatment of the theory of numerical series, i.e., infinite series whose terms are numbers. The picture is then completed by a third volume, entitled *Functional Series*, which, as its name implies, is devoted to the study of infinite series whose terms are functions. The set of three volumes makes up a comprehensive treatise on all aspects of a key topic of pure and applied mathematics.

As in the companion volume, the problems appearing at the end of each section constitute an important part of the course, and should not be neglected by the serious student.

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CHAPTER 1

Operations on Series

1. Associativity of Convergent Series

The concept of the sum of an infinite series differs substantially from the concept of the sum of a *finite* number of terms (considered in arithmetic and algebra) in that it involves taking a *limit*. In certain cases, properties of ordinary sums carry over to sums of infinite series, but usually only under certain conditions to be determined below. In other cases, the ordinary properties of finite series break down in a striking fashion. Thus, in general, we must be very careful when attempting to extend properties of finite sums to the case of infinite series.

Our first result along these lines expresses the *associativity* of convergent series, analogous to the similar property of finite sums:

THEOREM 1. *Let*

$$a_1 + a_2 + \cdots + a_n + \cdots \tag{1}$$

be a convergent series, and let[1]

$$\tilde{a}_1 + \tilde{a}_2 + \cdots + \tilde{a}_k + \cdots = (a_1 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + \cdots + (a_{n_{k-1}+1} + \cdots + a_{n_k}) + \cdots \tag{$\tilde{1}$}$$

1. Here $\{n_k\}$ is some subsequence of the sequence of positive integers 1, 2, ... such that $n_1 < n_2 < \cdots$.

be a new series obtained from (1) *by amalgamating the terms of* (1) *in arbitrary groups without changing the order of appearance of the terms. Then* $\tilde{(1)}$ *is convergent, with the same sum as* (1).

Proof. The sequence of partial sums

$$\begin{aligned}\tilde{A}_k &= \tilde{a}_1 + \tilde{a}_2 + \cdots + \tilde{a}_k \\ &= (a_1 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + \cdots \\ &\quad + (a_{n_{k-1}+1} + \cdots + a_{n_k}) \qquad (k = 1, 2, \ldots)\end{aligned}$$

of the new series $\tilde{(1)}$ is just the subsequence

$$A_{n_1}, A_{n_2}, \ldots, A_{n_k}, \ldots$$

of the sequence of partial sums

$$A_n = a_1 + \cdots + a_n \qquad (n = 1, 2, \ldots)$$

of the original series (1). But obviously every subsequence of a convergent sequence with limit A (say) is also convergent with limit A. ∎[2]

Unlike the case of finite sums, the associativity of convergent series breaks down if we try to apply the property in reverse order, so to speak. More exactly, suppose we start from a *convergent* series of the form $\tilde{(1)}$. Then the new series (1) obtained by dropping parentheses in $\tilde{(1)}$ may well turn out to be *divergent*. For example, the two series

$$(1 - 1) + (1 - 1) + (1 - 1) + \cdots = 0 + 0 + 0 \cdots = 0 \quad (2)$$

and

$$1 - (1 - 1) - (1 - 1) - \cdots = 1 - 0 - 0 - \cdots = 1 \quad (3)$$

both converge, but the series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

2. The symbol ∎ stands for Q.E.D. and indicates the end of a proof.

obtained by dropping parentheses in either (2) or (3) obviously diverges. However, Theorem 1 has the following qualified converse:

THEOREM 2. *Given a convergent series* $\tilde{(1)}$, *suppose the series* (1) *obtained by dropping parentheses in* $\tilde{(1)}$ *is convergent. Then its sum is the same as the sum of the original series* $\tilde{(1)}$.

Proof. An obvious consequence of Theorem 1. ∎

Under certain conditions, convergence of $\tilde{(1)}$ does in fact guarantee that of (1):

THEOREM 3. *Suppose the numbers inside each set of parentheses in the series* $\tilde{(1)}$ *all have the same sign, where this sign may vary from one set of parentheses to the next, and suppose* $\tilde{(1)}$ *is convergent. Then the series* (1) *obtained by dropping parentheses in* $\tilde{(1)}$ *is also convergent.*

Proof. In this case, the partial sum A_n either increases or decreases as n increases from n_{k-1} to n_k. Therefore

$$A_{n_{k-1}} \leqslant A_n \leqslant A_{n_k} \quad \text{or} \quad A_{n_{k-1}} \geqslant A_n \geqslant A_{n_k},$$

where $n = n_{k-1}, \ldots, n_k$. Equivalently,

$$\tilde{A}_{k-1} \leqslant A_n \leqslant \tilde{A}_k \quad \text{or} \quad \tilde{A}_{k-1} \geqslant A_n \geqslant \tilde{A}_k, \tag{4}$$

for the same values of n. But $\tilde{A}_k \to \tilde{A}$ as $k \to \infty$, where $\tilde{A}$ is the sum of $\tilde{(1)}$ and hence, by (4), $A_n \to \tilde{A}$ as $n \to \infty$. ∎

Example. Prove the convergence of the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{[\sqrt{k}]}}{k}, \tag{5}$$

where $[x]$ is the *integral part* of x, i.e., the largest integer $\leqslant x$.

Solution. Grouping together terms of identical sign in (5), we get

$$\sum_{k=1}^{\infty} (-1)^k \left[\frac{1}{k^2} + \frac{1}{k^2+1} + \cdots + \frac{1}{(k+1)^2 - 1} \right]$$
$$\equiv \sum_{k=1}^{\infty} (-1)^k c_k. \tag{6}$$

But

$$\frac{2}{2k+1} < \overbrace{\frac{1}{k^2} + \frac{1}{k^2+1} + \cdots + \frac{1}{k^2+k-1}}^{k \text{ terms}}$$

$$+ \overbrace{\frac{1}{k^2+k} + \cdots + \frac{1}{(k+1)^2-1}}^{k+1 \text{ terms}} < \frac{2}{k},$$

since each of the first k terms lies in the interval

$$\frac{1}{k^2+k} < x \leqslant \frac{1}{k^2},$$

while each of the last $k+1$ terms lies in the interval

$$\frac{1}{(k+1)^2} < x \leqslant \frac{1}{k^2+k}.$$

Therefore $\{c_k\}$ is a decreasing sequence with limit zero. It follows from Leibniz's test (*Ruds.*, p. 73)[3] that (6) converges. But then (5) also converges, by Theorem 3.

PROBLEM

Prove that if

$$p = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots,$$

$$q = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots,$$

then

$$p = \left(1 - \frac{1}{2^{s-1}}\right) q \tag{7}$$

if $s > 1$. Prove that (7) leads to an absurdity if $0 < s < 1$. Why?

3. The abbreviation *Ruds.* refers to the volume *Infinite Series: Rudiments* by G.M.Fichtenholz (The Pocket Mathematical Library).

2. Commutativity of Absolutely Convergent Series

Let

$$a_1 + a_2 + \cdots + a_n + \cdots \tag{1}$$

be a convergent series, and let

$$a_1' + a_2' + \cdots + a_k' + \cdots \equiv a_{n_1} + a_{n_2} + \cdots + a_{n_k} + \cdots \tag{1'}$$

be a new series obtained from (1) by rearranging its terms in any way whatsoever.[4] We then ask whether the series (1') converges, and if so, whether it has the same sum as the original series (1). There is now a sharp distinction between the case of *absolutely* convergent series and that of *conditionally* convergent series. In fact, absolutely convergent series, as opposed to conditionally convergent series (see Sec. 3), are *commutative* in the sense of the following

THEOREM. *Given an absolutely convergent series* (1), *the series* (1') *obtained by arbitrarily rearranging the terms of* (1) *is convergent, with the same sum as* (1).

Proof. First suppose the series (1) is positive. Given any positive integer k, let N be the largest of the integers $n_1, n_2, \ldots, n_k$. Then, obviously,

$$A_k' \leqslant A_N,$$

where A_N is the Nth partial sum of (1) and A_k' the kth partial sum of (1'). Hence, a fortiori,

$$A_k' \leqslant A,$$

where A is the sum of (1). It follows that (1') is convergent (why?), with sum A' satisfying the inequality

$$A' \leqslant A. \tag{2}$$

4. Thus $\{n_k\}$ is a sequence made up of *all* the positive integers $1, 2, \ldots$ (without omissions or duplications), but no longer arranged in increasing order.

On the other hand, the series (1) can also be regarded as a rearrangement of the terms of (1′), and hence

$$A \leqslant A', \tag{3}$$

by exactly the same argument. Comparing (2) and (3), we get

$$A' = A.$$

Now suppose (1) is an arbitrary series, whose terms can have values of either sign. The series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + \cdots + |a_n| + \cdots \tag{4}$$

converges, since (1) is absolutely convergent, and moreover, any rearrangement of (4) also converges, since (4) is a positive series. Therefore (1′) is also absolutely convergent, with sum A'. But the sum A of the absolutely convergent series (1) can be written in the form

$$A = P - Q,$$

where P and Q are the sums of the positive series

$$\sum_{j=1}^{\infty} p_j \tag{5}$$

and

$$\sum_{k=1}^{\infty} q_k, \tag{6}$$

formed from the positive terms $p_1, p_2, \ldots$ of (1) and the absolute values $q_1, q_2, \ldots$ of the negative terms (*Ruds.*, p. 64).[5] Rearranging the terms of (1) induces corresponding rearrangements of the terms of (5) and (6), but as shown in the first part of the proof, this has no effect on the sums P and Q of these series. It follows that

$$A' = P - Q = A. \quad \blacksquare$$

5. The terms of (5) and (6) are arranged in the same order as they (or their negatives) appear in (1).

PROBLEMS

1. Prove that the factors of an absolutely convergent infinite product (*Ruds.*, p. 97) can be rearranged arbitrarily without affecting either the convergence or the value of the product. How does this apply to the product

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right)$$

(*Ruds.*, p. 132)?

2. Verify the expansion

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x}{n\pi}\right)\left(1 + \frac{x}{n\pi}\right).$$

Prove that the factors on the right cannot be arbitrarily rearranged.

Hint. The series

$$-\frac{x}{\pi} + \frac{x}{\pi} - \frac{x}{2\pi} + \frac{x}{2\pi} - \cdots - \frac{x}{n\pi} + \frac{x}{n\pi} - \cdots$$

is conditionally convergent, and hence its terms cannot be arbitrarily rearranged (see Sec. 3).

3. Verify the expansion

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x}{n\pi}\right) e^{x/n\pi} \left(1 + \frac{x}{n\pi}\right) e^{-x/n\pi}.$$

Prove that the factors on the right can be arbitrarily rearranged.

Hint. The infinite product is now absolutely convergent (why?)

3. Riemann's Theorem

Next we show that a conditionally convergent series does not have the commutative property, and in fact can be made to have a different sum or even to diverge by suitably rearranging its terms.

LEMMA. *Let*

$$a_1 + a_2 + \cdots + a_n + \cdots \tag{1}$$

be a conditionally convergent series, let

$$p_1 + p_2 + \cdots + p_j + \cdots \tag{2}$$

be the series formed from the positive terms of (1), *and let*

$$q_1 + q_2 + \cdots + q_k + \cdots \tag{3}$$

be the series formed from the absolute values of the negative terms of (1).[6] *Then the series* (2) *and* (3) *both diverge.*

Proof. Let the series (1), (2) and (3) have partial sums A_n, P_j and Q_k, respectively. Then

$$A_n = P_j - Q_k \tag{4}$$

and

$$A_n^* = P_j + Q_k, \tag{5}$$

where A_n^* is the nth partial sum of the series

$$|a_1| + |a_2| + \cdots + |a_n| + \cdots, \tag{5*}$$

and j is the number of positive terms and k the number of negative terms in the first n terms of (1). If one of the series (2) and (3) converges, then so does the other, because of (4). But then the series (5*) converges, because of (5), i.e., (1) is absolutely convergent, contrary to hypothesis. This contradiction shows that (2) and (3) both diverge. ∎

Remark. Despite the fact that (2) and (3) both diverge, we have

$$\lim_{j \to \infty} p_j = \lim_{k \to \infty} q_k = 0, \tag{6}$$

6. Footnote 5 also applies here.

since (1) converges (conditionally) and hence

$$\lim_{n\to\infty} a_n = 0.$$

We can now prove the following remarkable

THEOREM **(Riemann's theorem).** *If the series* (1) *is conditionally convergent, then, given any number B (finite or equal to* $\pm\infty$*), the terms of* (1) *can be rearranged to give a new series whose sums equals B.*

Proof. First suppose B is finite. Since the series (2) and (3) both diverge, the same is true of all their remainders (*Ruds.*, Theorem 1, p. 6). Hence, starting *from any position* in the series (2) and (3), we can choose enough terms to make the sum *exceed any given number.* Using this fact, we now arrange the terms of the series (1) as follows. First we choose enough *positive* terms of (1), in their order of appearance in the series. to have a sum *greater* than B:

$$p_1 + p_2 + \cdots + p_{j_1} > B.$$

We then write down enough *negative* terms (again in their order of appearance in the series) to make the resulting sum *less* than B:

$$p_1 + p_2 + \cdots + p_{j_1} - q_1 - q_2 - \cdots - q_{k_1} < B.$$

Next we add more *positive* terms (from those that remain) to have

$$p_1 + \cdots + p_{j_1} - q_1 - \cdots - q_{k_1} + p_{j_1+1} + \cdots + p_{j_2} > B.$$

Then we choose enough negative terms (from those that remain) to have

$$p_1 + \cdots + p_{j_1} - q_1 - \cdots - q_{k_1}$$

$$+ p_{j_1+1} + \cdots + p_{k_2} - q_{k_1+1} - \cdots - q_{k_2} < B,$$

1 Fichtenholz (2094)

and so on. Obviously, every term in the original series will eventually appear in the new series if this process is continued indefinitely.

Now suppose that every time we write down groups of positive and negative terms, we see to it that no more terms are added than needed to make the sum greater than or less than B, as the case may be. Then the deviation of this sum from B (on one side or the other) can never have an absolute value greater than the last term written down. Therefore it follows from (6) that the new series

$$(p_1 + \cdots + p_{j_1}) - (q_1 + \cdots + q_{k_1}) + \cdots$$

$$+ (p_{j_{i-1}+1} + \cdots + p_{j_i}) + (q_{k_{i-1}+1} + \cdots + q_{k_i}) + \cdots \qquad (7)$$

is convergent, with sum B. Hence, by Theorem 3, p. 3, the sum of (7) after dropping parentheses is still B.

Finally suppose $B = +\infty$. Choosing an increasing sequence of numbers $B_1, B_2, \ldots$ approaching infinity, we first select enough positive terms to exceed B_1, afterwards adding one negative term, then select enough positive terms to exceed B_2, afterwards adding another negative term, and so on. This obviously gives a series with sum $+\infty$. Similarly, we can find a rearrangement of the series (1) with sum $-\infty$. ∎

Remark. Riemann's theorem makes it clear that conditional convergence of a series comes about only because of mutual cancellation of positive and negative terms, and hence depends in an essential way on the order of appearance of the terms. On the other hand, absolute convergence of a series comes about because of the rate at which the terms grow small, and does not depend on the order in which the terms appear.

Example. Consider the conditionally convergent series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2k-1} - \frac{1}{2k} + \cdots, \qquad (8)$$

with sum $\ln 2$ (*Ruds.*, p. 117). Rearranging the terms in such a way that two negative terms follow each positive term, we get

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \cdots$$
$$+ \frac{1}{2k-1} - \frac{1}{4k-2} - \frac{1}{4k} + \cdots \tag{9}$$

As we now show, the effect of this rearrangement is to make the sum of (8) one half as small. In fact, denoting the partial sums of (8) and (9) by A_n and A'_n, respectively, we have

$$\begin{aligned} A'_{3m} &= \sum_{k=1}^{m} \left(\frac{1}{2k-1} - \frac{1}{4k-2} - \frac{1}{4k} \right) \\ &= \sum_{k=1}^{m} \left(\frac{1}{4k-2} - \frac{1}{4k} \right) \\ &= \frac{1}{2} \sum_{k=1}^{m} \left(\frac{1}{2k-1} - \frac{1}{2k} \right) = \frac{1}{2} A_{2m}, \end{aligned}$$

and hence $A'_{3m} \to \frac{1}{2} \ln 2$. But

$$A'_{3m-1} = A'_{3m} + \frac{1}{4m}, \quad A'_{3m-2} = A'_{3m-1} + \frac{1}{4m-2}$$

approach the same limit $\frac{1}{2} \ln 2$. Therefore the series (10) is convergent, with sum $\frac{1}{2} \ln 2$.

PROBLEMS

1. Given a conditionally convergent series and any two numbers B and $C > B$, prove that the terms of (1) can be rearranged in such a way that the partial sums of the series have B as their smallest limit point (lower limit) and C as their largest limit point (upper limit).

2. Starting from the formula

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \ln n + C + \gamma_n$$

for the nth partial sum of the harmonic series (*Ruds.*, p. 18), where C is Euler's constant and $\gamma_n \to 0$ as $n \to \infty$, prove that

$$\overbrace{1 + \frac{1}{3} + \cdots + \frac{1}{2p-1}}^{p \text{ terms}} - \overbrace{\frac{1}{2} - \cdots - \frac{1}{2q}}^{q \text{ terms}} + \cdots$$

$$+ \overbrace{\frac{1}{2p+1} + \cdots + \frac{1}{4p-1}}^{p \text{ terms}}$$

$$- \overbrace{\frac{1}{2q+2} - \cdots - \frac{1}{4q}}^{q \text{ terms}} + \cdots = \ln\left(2\sqrt{\frac{p}{q}}\right),$$

where the series on the left is a rearrangement of the conditionally convergent series (8). In particular, verify that (8) and (9) have the sums $\ln 2$ and $\frac{1}{2}\ln 2$, as in the above example, and show that

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \frac{3}{2}\ln 2,$$

$$1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16} + \frac{1}{5} - \cdots = 0.$$

Is this kind of rearrangement sufficiently general to give a series with any preassigned sum whatsoever?

Hint. Note that

$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n} = \frac{1}{2}H_n = \frac{1}{2}\ln n + \frac{1}{2}C + \frac{1}{2}\gamma_n,$$

$$1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} = H_{2n} - \frac{1}{2}H_n$$

$$= \ln 2 + \frac{1}{2}\ln n$$

$$+ \frac{1}{2}C + \gamma_{2n} - \frac{1}{2}\gamma_n.$$

4. Multiplication of Series

We have already considered term-by-term addition (or subtraction) of two convergent series, as well as multiplication of a convergent series by a constant (*Ruds.*, p. 7). We now consider multiplication of two convergent series

$$A = \sum_{m=1}^{\infty} a_m = a_1 + a_2 + \cdots + a_m + \cdots \tag{1}$$

and

$$B = \sum_{n=1}^{\infty} b_n = b_1 + b_2 + \cdots + b_n + \cdots. \tag{2}$$

Guided by the rule for multiplying finite sums, we now consider all products of the form $a_j b_k$, arranged in an infinite "rectangular" matrix

$$\begin{array}{|l} \hline a_1b_1 \;\; a_2b_1 \;\; a_3b_1 \cdots a_jb_1 \cdots \\ a_1b_2 \;\; a_2b_2 \;\; a_3b_2 \cdots a_jb_2 \cdots \\ a_1b_3 \;\; a_2b_3 \;\; a_3b_3 \cdots a_jb_3 \cdots \\ \quad . \quad\;\; . \quad\;\; . \quad \cdots \quad . \quad \cdots \\ a_1b_k \;\; a_2b_k \;\; a_3b_k \cdots a_jb_k \cdots \\ \quad . \quad\;\; . \quad\;\; . \quad \cdots \quad . \quad \cdots \end{array} \tag{3}$$

There are many ways of arranging these products in a sequence. For example, we can enumerate products "by diagonals", indicated schematically by

$$\begin{array}{|l} \hline \quad\;\; \diagup \quad\;\; \diagup \quad\;\; \diagup \\ a_1b_1 \;\; a_2b_1 \;\; a_3b_1 \cdots \\ \diagup \quad\;\; \diagup \quad\;\; \diagup \quad\;\; \diagup \\ a_1b_2 \;\; a_2b_2 \;\; a_3b_2 \cdots \\ \quad \diagup \quad\;\; \diagup \quad\;\; \diagup \quad\;\; \diagup \\ a_1b_3 \;\; a_2b_3 \;\; a_3b_3 \cdots \\ \quad\;\; \diagup \quad\;\; \diagup \quad\;\; \diagup \\ \quad . \quad\;\; . \quad\;\; . \quad \cdots \end{array} \tag{4}$$

or "by squares," indicated by

$$\begin{array}{|c|c|c|c|} \hline a_1b_1 & a_2b_1 & a_3b_1 & \cdots \\ \hline a_1b_2 & a_2b_2 & a_3b_2 & \cdots \\ \hline a_1b_3 & a_2b_3 & a_3b_3 & \cdots \\ \hline . & . & . & \cdots \\ \hline \end{array} \tag{3}$$

This gives the sequences

$$a_1b_1, a_1b_2, a_2b_1, a_1b_3, a_2b_2, a_3b_1, \ldots \tag{4'}$$

and

$$a_1b_1, a_1b_2, a_2b_2, a_2b_1, a_1b_3, a_2b_3, a_3b_3, a_3b_2, a_3b_1, \ldots, \tag{5'}$$

respectively. A series formed from a sequence like (4') or (5') is called a *product* of the series (1) and (2). Every such product has the same value if (1) and (2) are *absolutely* convergent, as shown by the following

THEOREM **(Cauchy's theorem).** *If two series* (1) *and* (2) *are absolutely convergent, with sums A and B, then their product, formed from the numbers* (3) *taken in any order, is also absolutely convergent, with sum AB.*

Proof. By hypothesis, the series

$$\sum_{m=1}^{\infty} |a_m| = |a_1| + |a_2| + \cdots + |a_m| + \cdots \tag{1*}$$

and

$$\sum_{n=1}^{\infty} |b_n| = |b_1| + |b_2| + \cdots + |b_n| + \cdots \tag{2*}$$

both converge, i.e., have finite sums A^* and B^*, say. Making an arbitrary arrangement of the products (3), we get a series

$$\sum_{s=1}^{\infty} a_{j_s}b_{k_s} = a_{j_1}b_{k_1} + a_{j_2}b_{k_2} + \cdots + a_{j_s}b_{k_s} + \cdots. \tag{6}$$

To prove the absolute convergence of the series (6), we estimate its partial sums. Let N be the largest of the integers $j_1, k_1, j_2,$

$k_2, \ldots, j_s, k_s$. Then obviously

$$\begin{aligned} &|a_{j_1}b_{k_1}| + |a_{j_2}b_{k_2}| + \cdots + |a_{j_s}b_{k_s}| \\ &\quad \leqslant (|a_1| + |a_2| + \cdots + |a_N|)(|b_1| + |b_2| + \cdots + |b_N|) \\ &\quad \leqslant A^*B^*. \end{aligned} \tag{7}$$

Since (7) holds for all s, the series (6) converges absolutely, as asserted.

To determine the sum of (6), we rely on the absolute convergence of the series and Riemann's theorem to arrange the terms of (6) more conveniently. Enumerating these terms "by squares," as in (5), we amalgamate the groups of terms by which consecutive squares differ from each other, obtaining

$$\begin{aligned} &a_1b_1 + (a_1b_2 + a_2b_2 + a_2b_1) \\ &\quad + (a_1b_3 + a_2b_3 + a_3b_3 + a_3b_2 + a_3b_1) + \cdots. \end{aligned} \tag{8}$$

Then, letting A_m and B_n denote the partial sums of (1) and (2), we find that (8) has partial sums

$$\begin{aligned} A_1B_1 &= a_1b_1, \\ A_2B_2 &= (a_1 + a_2)(b_1 + b_2) \\ A_3B_3 &= (a_1 + a_2 + a_3)(b_1 + b_2 + b_3), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

which clearly converge to the product AB. But AB is the sum of (6), as well as of (8). ∎

Remark. In actually carrying out multiplication of series, it is usually best to arrange the products (3) "by diagonals," as in (4), and then amalgamate terms lying on each diagonal:

$$AB = a_1b_1 + (a_1b_2 + a_2b_1) + (a_1b_3 + a_2b_2 + a_3b_1) + \cdots. \tag{9}$$

This is the form in which Cauchy first represented the product of two series. Therefore (9) will henceforth be called the product of the series (1) and (2) in *Cauchy's form.*

Example. Using (9) to multiply two power series

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m + \cdots,$$
$$\sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots \tag{10}$$

(where x lies within the intervals of convergence of both series), we find that

$$\sum_{m=0}^{\infty} a_m x^m \sum_{n=0}^{\infty} b_n x^n$$
$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)\, x + (a_0 b_2 + a_1 b_1 + a_2 b_0)\, x^2 + \cdots. \tag{11}$$

Thus the product of two power series in Cauchy's form is automatically represented in the form of a new power series. The process leading from (10) to (11) is often described as "term-by-term multiplication" (in accordance with *Cauchy's rule*).

PROBLEMS

1. By multiplying the series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots \qquad (|x| < 1)$$

by itself, prove that

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + \cdots + n x^{n-1} + \cdots$$
$$(|x| < 1).$$

2. By multiplying the series

$$\frac{1}{1+x} = \sum_{m=0}^{\infty} (-1)^m x^m = 1 - x + x^2 - \cdots + (-1)^m x^m + \cdots$$
$$(|x| < 1)$$

by the series

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots \qquad (|x| < 1)$$

(*Ruds.*, p. 117), prove that

$$\frac{\ln(1+x)}{1+x} = \sum_{n=1}^{\infty} (-1)^n H_n x^n = x - \left(1 + \frac{1}{2}\right) x^2 + \cdots$$

$$+ (-1)^n \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) x^n + \cdots \qquad (|x| < 1),$$

where H_n is the nth partial sum of the harmonic series.

3. Use term-by-term multiplication to prove the identity

$$\sum_{n=0}^{\infty} A_n x^n = \frac{1}{1-x} \sum_{n=0}^{\infty} a_n x^n$$

or

$$\sum_{n=0}^{\infty} a_n x^n = (1-x) \sum_{n=0}^{\infty} A_n x^n,$$

where $A_n = a_0 + a_1 + \cdots + a_n$ (for another proof, see *Ruds.*, Prob. 1, p. 85).

Comment. If one of the two series converges in the interval $(-R, R)$, where $0 < R \leqslant 1$, then so does the other (why?).

4. Prove the identity

$$\left(\frac{1}{a} + \frac{1}{2}\frac{x}{a+2} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^2}{a+4} + \cdots\right) \times$$

$$\times \left(1 + \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 + \cdots\right)$$

$$= \frac{1}{a}\left(1 + \frac{a+1}{a+2}x + \frac{(a+1)(a+3)}{(a+2)(a+4)}x^2 + \cdots\right),$$

where $a > 0$.

5. Suppose it were not known in advance that the exponential function obeys the "addition formula"

$$e^x e^y = e^{x+y}. \tag{12}$$

Prove that (12) follows from term-by-term multiplication of the series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

(*Ruds.*, p. 113) by the same series with x replaced by y.

Hint. Use the binomial theorem.

6. Suppose it were not known in advance that the sine and cosine functions obey the "addition formulas"

$$\sin(x + y) = \sin x \cos y + \cos x \sin y,$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y.$$

Prove that these formulas follow from term-by-term multiplication of the series

$$\sin x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \cdots,$$

$$\cos x = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

(*Ruds.*, p. 114), with appropriate changes of x to y.

7. Find the square of the series

$$1 + \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n}(n!)^2} \quad (x \text{ arbitrary}).$$

Hint. First prove the combinatorial formula

$$\sum_{k=0}^{n} (C_k^n)^2 = C_n^{2n},$$

involving the binomial coefficients

$$C_k^n = \frac{n!}{k!\,(n-k)!}, \quad C_n^{2n} = \frac{(2n)!}{(n!)^2}.$$

Ans.

$$1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2n)!\, x^{2n}}{2^{2n}(n!)^4}.$$

8. Square the series

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \quad (x > 1),$$

representing the Riemann zeta function (*Ruds.*, pp. 11, 45).

Ans. We have

$$[\zeta(x)]^2 = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^x},$$

where $\tau(n)$ is the number of positive integers dividing n.

9. Let $\zeta(x)$ be the same as in the preceding problem. Prove that

$$\zeta(x) = \prod_{k=1}^{\infty} \frac{1}{1 - \frac{1}{p_k^x}},$$

where $\{p_k\}$ is the sequence of all prime numbers greater than 1, arranged in increasing order.[7]

7. For example, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, ...

Hint. Clearly

$$\frac{1}{1-\frac{1}{p_k^x}} = 1 + \frac{1}{p_k^x} + \frac{1}{p_k^{2x}} + \cdots + \frac{1}{p_k^{mx}} + \cdots,$$

and hence

$$P_x^{(N)} = \prod_{p_k \leqslant N} \frac{1}{1-\frac{1}{p_k^x}} = \sum_{n=1}^{\infty}{}' \frac{1}{n^x} = \sum_{n=1}^{N} \frac{1}{n^x} + \sum_{n=N+1}^{\infty}{}' \frac{1}{n^x}, \quad (13)$$

where the prime shows that the summation does not extend over all positive integers, but only over those (except for 1) containing the numbers $p_1, p_2, \ldots, p_k \leqslant N$ as factors. Therefore

$$0 < P_x^{(N)} - \sum_{n=1}^{N} \frac{1}{n^x} < \sum_{n=N+1}^{\infty} \frac{1}{n^x},$$

a fortiori. But the right-hand side approaches 0 as $N \to \infty$ (why?), and hence

$$\lim_{N\to\infty} P_x^{(N)} = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

10. Prove that

$$\prod_{k=1}^{\infty} \frac{1}{1-\frac{1}{p_k}} = \frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{p_k}\right)} = +\infty. \quad (14)$$

Deduce from this that there are infinitely many primes.

Hint. Formula (13) continues to hold if $x = 1$, and hence

$$P_1^{(N)} = \prod_{k \leqslant N} \frac{1}{1-\frac{1}{p_k}} > \sum_{n=1}^{N} \frac{1}{n} = H_n.$$

11. Prove that the series

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{p_k}+\cdots=\sum_{k=1}^{\infty}\frac{1}{p_k} \tag{15}$$

diverges.

Hint. Since the product (14) diverges, the series (15) also diverges (*Ruds.*, Theorem 5, p. 96).

Comment. This serves to characterize the rate of growth of the prime numbers, and is clearly a much stronger result than the fact that the harmonic series diverges.

12. Prove that

$$\frac{1}{\left(1+\frac{1}{3^x}\right)\left(1-\frac{1}{5^x}\right)\left(1+\frac{1}{7^x}\right)\left(1+\frac{1}{11^x}\right)\cdots\left(1\pm\frac{1}{p_{k+1}^x}\right)\cdots}$$

$$=1-\frac{1}{3^x}+\frac{1}{5^x}-\frac{1}{7^x}+\frac{1}{9^x}-\cdots \qquad (x<1),$$

where we use a plus or minus sign in the denominator of the expression on the left depending on whether the factor in question involves a prime of the form $4n-1$ or of the form $4n+1$.

5. Toeplitz's Theorem

We now prove a general limit theorem which will be needed later on (in Sec. 6 and Chap. 4):

THEOREM **(Toeplitz's theorem).** *Given an infinite "triangular" matrix*

$$\begin{array}{|llllll}
t_{11} & & & & & \\
t_{21} & t_{22} & & & & \\
t_{31} & t_{32} & t_{33} & & & \\
\cdot & \cdot & \cdot & & & \\
t_{n1} & t_{n2} & t_{n3} & \cdots & t_{nn} & \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdots
\end{array}$$

of numbers t_{nm} $(1 \leqslant m \leqslant n)$, *suppose that the elements in each column converge to zero, i.e.,*

$$t_{nm} \to 0 \quad as \quad n \to \infty \qquad (m\ fixed), \tag{1}$$

while the sums of the absolute values of the elements in each row are all bounded by the same constant K:

$$|t_{n1}| + |t_{n2}| + \cdots + |t_{nn}| \leqslant K \qquad (n = 1, 2, \ldots). \tag{2}$$

Let $\{x_n\}$ *be any sequence converging to zero, and let* $\{x_n'\}$ *be the sequence with general term*

$$x_n' = t_{n1}x_1 + t_{n2}x_2 + \cdots + t_{nn}x_n.$$

Then $\{x_n'\}$ *also converges to zero.*

Proof. Given any $\varepsilon > 0$, there is an m such that

$$|x_n| < \frac{\varepsilon}{2K} \quad \text{if} \quad n > m,$$

and hence, by (2)

$$\begin{aligned} |x_n'| &\leqslant |t_{n1}x_1 + \cdots + t_{nm}x_m| + |t_{n,m+1}x_{m+1} + \cdots + t_{nn}x_n| \\ &< |t_{n1}x_1 + \cdots + t_{nm}x_m| + (|t_{n,m+1}| + \cdots + |t_{nn}|)\,\frac{\varepsilon}{2K} \\ &\leqslant |t_{n1}x_1 + \cdots + t_{nm}x_m| + \frac{\varepsilon}{2} \quad \text{if} \quad n > m. \end{aligned} \tag{3}$$

But m is fixed in (3), and hence, by (1), there is an $N \geqslant m$ such that

$$|t_{n1}x_1 + \cdots + t_{nm}x_m| < \frac{\varepsilon}{2} \quad \text{if} \quad n > N. \tag{4}$$

Combining (3) and (4), we find that

$$|x_n'| < \varepsilon \quad \text{if} \quad n > N,$$

i.e., $x_n' \to 0$ as $n \to \infty$. ∎

COROLLARY 1. *Suppose the coefficients* t_{nm} *satisfy the condition*[8]

$$T_n = t_{n1} + t_{n2} + \cdots + t_{nn} \to 1$$

as $n \to \infty$, *in addition to the conditions* (1) *and* (2), *and suppose*

$$\lim_{n\to\infty} x_n = a$$

(*a finite*). *Then*

$$\lim_{n\to\infty} x_n' = \lim_{n\to\infty} (t_{n1}x_1 + t_{n2}x_2 + \cdots + t_{nn}x_n) = a.$$

Proof. Clearly

$$x_n' = t_{n1}(x_1 - a) + t_{n2}(x_2 - a) + \cdots + t_{nn}(x_n - a) + T_n a,$$

where, by hypothesis, $x_n - a \to 0$ as $n \to \infty$. Hence, by Toeplitz's theorem,

$$\lim_{n\to\infty} x_n' = \lim_{n\to\infty} T_n a = a. \quad \blacksquare$$

COROLLARY 2. *If*

$$\lim_{n\to\infty} x_n = a,$$

then

$$\lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} = a.$$

Proof. Choose

$$t_{n1} = t_{n2} = \cdots = t_{nn} = \frac{1}{n}$$

in Corollary 1. $\blacksquare$

COROLLARY 3. *Suppose*

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0,$$

where

$$|y_1| + |y_2| + \cdots + |y_n| \leqslant K \qquad (n = 1, 2, \ldots)$$

8. In the applications, we usually have $T_n \equiv 1$.

for a suitable constant K, and let

$$z_n = x_1 y_n + x_2 y_{n-1} + \cdots + x_n y_1 .$$

Then

$$\lim_{n\to\infty} z_n = 0.$$

Proof. Choose $t_{nm} = y_{n-m+1}$ in Toeplitz's theorem. ∎

COROLLARY 4. *Suppose*

$$\lim_{n\to\infty} x_n = a, \quad \lim_{n\to\infty} y_n = b,$$

and let

$$z_n = \frac{x_1 y_n + x_2 y_{n-1} + \cdots + x_n y_1}{n}.$$

Then

$$\lim_{n\to\infty} z_n = ab.$$

Proof. First let $a = 0$. Then $z_n \to 0$, by Corollary 3, if we replace y_n by y_n/n, noting that $|y_n| \leqslant K$ (why does K exist?) implies

$$\left|\frac{y_1}{n}\right| + \left|\frac{y_2}{n}\right| + \cdots + \left|\frac{y_n}{n}\right| \leqslant \frac{nK}{n} = K.$$

If $a \neq 0$, we write

$$z_n = \frac{(x_1 - a)\, y_n + (x_2 - a)\, y_{n-1} + \cdots + (x_n - a)\, y_1}{n}$$

$$+ a\,\frac{y_1 + y_2 + \cdots + y_n}{n}.$$

Then the first term on the right approaches zero, for the reason just given, while the second approaches ab, by Corollary 2. ∎

PROBLEMS

1. Given two sequences $\{x_n\}$ and $\{y_n\}$, suppose that

a) $\{y_n\}$ is an increasing sequence with limit $+\infty$;

b) $$\lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = a \qquad (x_0 = y_0 = 0).$$

Prove that

$$\lim_{n\to\infty} \frac{x_n}{y_n} = a.$$

Hint. Choose

$$t_{nm} = \frac{y_m - y_{m-1}}{y_n}.$$

2. Given a sequence $x_0, x_1, x_2, \ldots, x_n, \ldots$, let

$$x_n' = \frac{C_0^n x_0 + C_1^n x_1 + C_2^n x_2 + \cdots + C_n^n x_n}{2^n},$$

where C_m^n is the binomial coefficient

$$C_m^n = \frac{n!}{m!\,(n-m)!} \qquad (m = 0, 1, \ldots, n).$$

Prove that $x_n \to a$ implies $x_n' \to a$.

Hint. Let

$$t_{nm} = \frac{C_m^n}{2^n},$$

noting that $C_m^n < n^m$ and hence $t_{nm} \to 0$ as $n \to \infty$, while

$$\sum_{m=0}^{n} C_m^n = 2^n.$$

3. Given a sequence $x_0, x_1, x_2, \ldots, x_n, \ldots$, let

$$x_n' = \frac{C_0^n x_0 + C_1^n x_1 z + C_2^n x_2 z^2 + \cdots + C_n^n x_n z^n}{(1+z)^n},$$

$$x_n'' = \frac{C_0^n x_0 z^n + C_1^n x_1 z^{n-1} + C_2^n x_2 z^{n-2} + \cdots + C_n^n x_n}{(1+z)^n},$$

where $z > 0$. Prove that $x_n \to a$ implies $x_n' \to a$, $x_n'' \to a$.

6. The Theorems of Mertens and Abel

We are now in a position to prove a theorem generalizing Cauchy's theorem on the multiplication of series:

THEOREM 1 **(Merten's theorem).** *Given two convergent series*

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots \tag{1}$$

and

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + \cdots + b_n + \cdots, \tag{2}$$

with sums A and B, respectively, suppose at least one of the series is absolutely convergent. Then the product of (1) *and* (2) *in Cauchy's form is convergent, with sum AB, i.e.,*

$$AB = a_1b_1 + (a_1b_2 + a_2b_1) + (a_1b_3 + a_2b_2 + a_3b_1) + \cdots. \tag{3}$$

Proof. Suppose the first series (say) is absolutely convergent, i.e., suppose the series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + \cdots + |a_n| + \cdots \tag{4}$$

is convergent. Combining terms on the nth diagonal of the matrix (3), p. 13 we write

$$c_n = a_1b_n + a_2b_{n-1} + \cdots + a_{n-1}b_2 + a_nb_1,$$

so that the series on the right in (3) is just

$$c_1 + c_2 + \cdots + c_n + \cdots,$$

with partial sums

$$C_n = c_1 + c_2 + \cdots + c_n \qquad (n = 1, 2, \ldots).$$

Thus our aim is to show that $C_n \to AB$ as $n \to \infty$. First of all, it is easy to see that

$$C_n = a_1B_n + a_2B_{n-1} + \cdots + a_{n-1}B_2 + a_nB_1,$$

where B_n is the nth partial sum of the series (2). Setting $B_n = B - \beta_n$ (where the remainder $\beta_n \to 0$ as $n \to \infty$), we can write C_n in the form

$$C_n = A_n B - \gamma_n,$$

where A_n is the nth partial sum of the series (1) and

$$\gamma_n = a_1\beta_n + a_2\beta_{n-1} + \cdots + a_{n-1}\beta_2 + a_n\beta_1.$$

Since $A_n \to A$, everything reduces to showing that

$$\lim_{n\to\infty} \gamma_n = 0.$$

But this follows at once from Corollary 3, p. 23 (with $x_n = \beta_n$, $y_n = a_n$), if we bear in mind that

$$|a_1| + |a_2| + \cdots + |a_n| \leqslant A^* \qquad (n = 1, 2, \ldots),$$

where A^* is the sum of the series (4). ∎

Example 1. It cannot be asserted that the series on the right in (3) converges if both series (1) and (2) are only conditionally convergent. For example, suppose we try to multiply the conditionally convergent series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + (-1)^{n-1}\frac{1}{\sqrt{n}} + \cdots$$

(*Ruds.*, Prob. 1a, p. 76) by itself. This gives

$$c_n = (-1)^{n-1}\left(\frac{1}{1\cdot\sqrt{n}} + \frac{1}{\sqrt{2}\sqrt{n-1}} + \cdots + \frac{1}{\sqrt{k}\sqrt{n-k+1}} + \cdots + \frac{1}{\sqrt{n}\cdot 1}\right).$$

But $|c_n| > 1$, since

$$\frac{1}{\sqrt{k}\sqrt{n-k+1}} > \frac{1}{n} \qquad (k = 1, \ldots, n),$$

and hence

$$\sum_{n=1}^{\infty} c_n$$

diverges.

Example 2. On the other hand, suppose we multiply the conditionally convergent series

$$\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^{n-1}\frac{1}{n} + \cdots$$

(*Ruds.*, p. 117) by itself. This gives

$$c_n = (-1)^{n-1} \times$$

$$\times \left(\frac{1}{1 \cdot n} + \frac{1}{2(n-1)} + \cdots + \frac{1}{k(n-k+1)} + \cdots + \frac{1}{n \cdot 1} \right)$$

$$= (-1)^{n-1} \sum_{k=0}^{n} \frac{1}{n(n-k+1)}$$

$$= \frac{(-1)^{n-1}}{n+1} \sum_{k=0}^{n} \left(\frac{1}{k} + \frac{1}{n-k+1} \right)$$

$$= (-1)^{n-1} \frac{2}{n+1} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} \right).$$

Thus $\{|c_n|\}$ is a decreasing sequence converging to zero (why?), and hence

$$\sum_{n=1}^{\infty} c_n \tag{5}$$

converges by Leibniz's test (*Ruds.*, p. 73). It is now natural to ask whether the sum of (5) is $(\ln 2)^2$. The answer is affirmative, as shown by the next theorem.

THEOREM 2 **(Abel's theorem).** *Given two convergent series* (1) *and* (2), *with sums A and B, respectively, suppose their product*

$$\sum_{n=1}^{\infty} c_n = c_1 + c_2 + \cdots + c_n + \cdots$$

is convergent, with sum C. Then

$$C = AB.$$

Proof. With the same notation as before, we easily find that

$$C_1 + C_2 + \cdots + C_n = A_1B_n + A_2B_{n-1} + \cdots + A_nB_1.$$

It follows that

$$\lim_{n\to\infty} \frac{C_1 + C_2 + \cdots + C_n}{n}$$

$$= \lim_{n\to\infty} \frac{A_1B_n + A_2B_{n-1} + \cdots + A_nB_{n-1}}{n}.$$

But the left-hand side equals C, by Corollary 2, p. 23, while the right-hand side equals AB, by Corollary 4, p. 24 (with $x_n = A_n$, $y_n = B_n$). ∎

PROBLEM

Suppose the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

has the interval of convergence $(-R, R)$, where $R < 1$, and suppose the series converges (possibly only conditionally) at the end points $x = \pm R$. Prove that the identity in Problem 3, p. 17 continues to hold for $x = \pm R$.

CHAPTER 2

Iterated and Double Series

7. Iterated Series

Given infinitely many real numbers

$$a_j^{(k)} \qquad (j, k = 1, 2, \ldots)$$

indexed by two integers j and k, imagine these numbers arranged in an infinite rectangular matrix

$$\begin{vmatrix} a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & \cdots & a_j^{(1)} & \cdots \\ a_1^{(2)} & a_2^{(2)} & a_3^{(2)} & \cdots & a_j^{(2)} & \cdots \\ a_1^{(3)} & a_2^{(3)} & a_3^{(3)} & \cdots & a_j^{(3)} & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdots \\ a_1^{(k)} & a_2^{(k)} & a_3^{(k)} & \cdots & a_j^{(k)} & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdots \end{vmatrix} \tag{1}$$

Suppose we first sum the elements in each row of (1) separately, obtaining an infinite sequence of series of the form

$$\sum_{j=1}^{\infty} a_j^{(k)} \qquad (k = 1, 2, \ldots), \tag{2}$$

and then sum the terms of this sequence in turn, obtaining the expression

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_j^{(k)}, \tag{3}$$

called an *iterated series.* Alternatively, if we sum the matrix (1) first over columns and then over rows, we get another kind of iterated series, of the form

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_j^{(k)}. \tag{3'}$$

DEFINITION 1. *Suppose summation of the elements in the kth row of* (1) *gives a convergent series, with sum* $A^{(k)}$, *for every* $k = 1, 2, \ldots$, *and suppose the series*

$$\sum_{k=1}^{\infty} A^{(k)} \tag{4}$$

is convergent, with sum A. Then the iterated series (3) *is said to be* **convergent,** *with* **sum** *A.*

DEFINITION 1'. *Suppose summation of the elements in the jth column of* (1) *gives a convergent series, with sum* A_j, *for every* $j = 1, 2, \ldots$, *and suppose the series*

$$\sum_{j=1}^{\infty} A_j \tag{4'}$$

is convergent, with sum A'. Then the iterated series (3') *is said to be* **convergent,** *with* **sum** *A'.*

The elements of the matrix (1) can be represented as an ordinary sequence

$$u_1, u_2, \ldots, u_r, \ldots \tag{5}$$

in infinitely many ways, and there is an ordinary series

$$\sum_{r=1}^{\infty} u_r \tag{6}$$

corresponding to each such sequence.[1] Conversely, given any ordinary sequence (5), the terms of the sequence can be arranged in infinitely many ways (without regard for order) to make up a

1. A similar situation has already been encountered in connection with the matrix of the special form (3), p. 13.

matrix of the form (1), and there is an inerated series of the form (3), say, corresponding to each such arrangement. Thus it is natural to ask about the relation between the series (3) and (6). This is done in the following two theorems:

THEOREM 1. *If the series* (6) *is absolutely convergent, with sum* U, *then, no matter how its terms are arranged as a matrix of the form* (1), *the corresponding iterated series* (3) *is convergent, with the same sum* U.

Proof. By hypothesis, the series

$$\sum_{r=1}^{\infty} |u_r| \tag{6*}$$

is convergent. Let U^* be its sum. Then, given any k and n,

$$\sum_{j=1}^{n} |a_j^{(k)}| \leqslant U^*.$$

Therefore

$$\sum_{j=1}^{\infty} |a_j^{(k)}|$$

converges, and hence so does

$$\sum_{j=1}^{\infty} a_j^{(k)},$$

since an absolutely convergent series is always convergent (*Ruds.*, Theorem 1, p. 63). Moreover, given any $\varepsilon > 0$, there is an integer r_0 such that

$$\sum_{r=r_0+1}^{\infty} |u_r| < \varepsilon, \tag{7}$$

and hence, *a fortiori*,

$$\left| \sum_{r=r_0+1}^{\infty} u_r \right| = \left| U - \sum_{r=1}^{r_0} u_r \right| < \varepsilon. \tag{8}$$

The terms $u_1, u_2, \ldots, u_{r_0}$ of the series (6) all belong to the first m columns and the first n rows of the matrix (1) if m and n are

sufficiently large, say if $m > m_0$, $n > n_0$. Therefore, by (7),

$$\left| \sum_{k=1}^{n} \sum_{j=1}^{m} a_j^{(k)} - \sum_{r=1}^{r_0} u_r \right| < \varepsilon \qquad (m > m_0, n > n_0),$$

since the expression whose absolute value appears on the left is a group of terms u_r with indices greater than r_0. Taking the limit as $m \to \infty$, we get

$$\left| \sum_{k=1}^{n} A^{(k)} - \sum_{r=1}^{r_0} u_r \right| < \varepsilon \qquad (n > n_0),$$

which, together with (8), implies

$$\left| \sum_{k=1}^{n} A^{(k)} - U \right| < 2\varepsilon,$$

and hence

$$\sum_{k=1}^{\infty} A^{(k)} = \lim_{n \to \infty} \sum_{k=1}^{n} A^{(k)} = U. \quad \blacksquare$$

Remark. Theorem 1 obviously continues to hold in the case where some of the rows of the matrix (1) contain only finitely many terms.

DEFINITION 2. *The iterated series* (3) *is said to be* **absolutely convergent** *if the iterated series*

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |a_j^{(k)}|, \tag{9}$$

whose terms are the absolute values of those of (3), *is convergent.*

Remark. Similarly, the iterated series (3′) is said to be absolutely convergent if the series

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |a_j^{(k)}|$$

is convergent.

THEOREM 2. *Suppose the iterated series* (3) *is absolutely convergent. Then* (3) *is convergent, and the series* (6) *made up of the elements of* (1) *arranged in any order whatsoever is absolutely convergent, with the same sum as* (3).

By hypothesis, the series (9) is convergent. Let A^* be its sum. Then, given any m and n, we have

$$\sum_{k=1}^{n} \sum_{j=1}^{m} |a_j^{(k)}| \leqslant A^*. \tag{10}$$

Consider any partial sum

$$U_r^* = |u_1| + |u_2| + \cdots + |u_r|$$

of the series (6*). The numbers $u_1, u_2, \ldots, u_r$ all belong to the first m columns and the first n rows of the matrix (1) if m and n are sufficiently large. Hence (10) implies

$$U_r^* \leqslant A^* \quad (r = 1, 2, \ldots),$$

so that (6*) is convergent, i.e., (6) is absolutely convergent. The rest of the proof is now an immediate consequence of Theorem 1. ∎

Remark. Clearly, Theorems 1 and 2 remain true if (3) is replaced by (3′).

THEOREM 3. *Given a matrix* (1), *suppose the iterated series* (3) *is absolutely convergent. Then the iterated series* (3′) *converges and has the same sum as* (3):

$$\sum_{j=1}^{m} \sum_{k=1}^{n} a_j^{(k)} = \sum_{k=1}^{n} \sum_{j=1}^{m} a_j^{(k)}.$$

Proof. By Theorem 2, the series made up of the elements of (1) arranged in any order whatsoever is absolutely convergent. Hence, by Theorem 1, the iterated series (3) corresponding to the matrix (1) as it stands and the iterated series (3′), which is of the form (3) if we interchange rows and columns in the matrix (1), both converge and have the same sum. ∎

Remark. Clearly Theorem 3 remains true if we interchange the roles of (3) and (3′).

PROBLEM

Interpret Theorem 1 as a generalization of both Theorem 1, p. 1 and the theorem on p. 5.

8. Double Series

There is another kind of series associated with the infinite rectangular matrix (1), p. 30, namely the series of the form

$$\begin{aligned} &a_1^{(1)} + a_2^{(1)} + \cdots + a_j^{(1)} + \cdots \\ &+ a_1^{(2)} + a_2^{(2)} + \cdots + a_j^{(2)} + \cdots \\ &+ \cdots\cdots \\ &+ a_1^{(k)} + a_2^{(k)} + \cdots + a_j^{(k)} + \cdots \\ &+ \cdots\cdots, \end{aligned}$$

called a *double series* and denoted by

$$\sum_{j,k=1}^{\infty} a_j^{(k)} \tag{1}$$

(with only one summation sign). By a *partial sum* of (1), we mean a finite sum of the form

$$A_m^{(n)} = \sum_{j=1}^{m} \sum_{k=1}^{n} a_j^{(k)},$$

made up of the terms in the first m columns and the first n rows of the given matrix. Let m and n approach infinity *independently*. Then the limit[2]

$$A = \lim_{\substack{m\to\infty \\ n\to\infty}} A_m^{(n)} \tag{2}$$

2. The limit in (2) is a *double limit*. Thus if A is finite, (2) means that, given any $\varepsilon > 0$, there is an integer N such that

$$|A_m^{(n)} - A| < \varepsilon$$

whenever m and n both exceed N.

(finite or infinite) is called the *sum* of the double series (1), and we write

$$A = \sum_{j,k=1}^{\infty} a_j^{(k)}.$$

The series (1) is said to be *convergent* if it has a finite sum, and *divergent* otherwise.

It is now natural to compare the double series (1) with the iterated series

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_j^{(k)} \tag{3}$$

and

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_j^{(k)}. \tag{3'}$$

Since

$$A_m^{(n)} = \sum_{k=1}^{n} \sum_{j=1}^{m} a_j^{(k)},$$

we have

$$\lim_{m\to\infty} A_m^{(n)} = \sum_{k=1}^{n} A^{(k)},$$

after taking the limit as $m \to \infty$ for fixed n. Here

$$A^{(k)} = \sum_{j=1}^{\infty} a_j^{(k)} \quad (k = 1, 2, \ldots), \tag{4}$$

and every "row series" on the right is assumed to converge. Thus the sum of the iterated series (3) is just the *iterated limit*

$$\lim_{n\to\infty} \lim_{m\to\infty} A_m^{(n)}.$$

Similarly, the sum of (3') is the other iterated limit

$$\lim_{m\to\infty} \lim_{n\to\infty} A_m^{(n)}$$

(give the details).

THEOREM 1. *If the double series* (1) *converges and if every "row series"* (4) *converges, then the iterated series* (3) *also converges and*

has the same sum as the double series:

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_j^{(k)} = \sum_{j,k=1}^{\infty} a_j^{(k)}. \tag{5}$$

Proof. Given any $\varepsilon > 0$, there is an integer N such that

$$|A_m^{(n)} - A| < \varepsilon,$$

provided that $m, n > N$. Holding $n > N$ fixed, we take the limit as $m \to \infty$, obtaining

$$\left| \sum_{k=1}^{n} A^{(k)} - A \right| \leqslant \varepsilon.$$

Taking the limit as $n \to \infty$ now gives

$$\left| \sum_{k=1}^{\infty} A^{(k)} - A \right| \leqslant \varepsilon,$$

which implies (5). ∎

Remark. There is an obvious analogue of Theorem 1 for the case of the iterated series (3′).

DEFINITION 1. *A double series* (1) *such that*

$$a_j^{(k)} \geqslant 0 \qquad (j, k = 1, 2, \ldots)$$

is said to be **positive.**

For such series we have the exact analogue of a familiar result for ordinary positive series (*Ruds.*, Theorem, p. 9):

THEOREM 2. *A positive double series always has a sum. The sum is finite, and hence the series converges, if the partial sums of the series are bounded from above. Otherwise the sum is infinite, and hence the series diverges.*

Proof. The last assertion is obvious. To prove that a positive double series (1) converges if its partial sums are bounded, suppose

$$A_m^{(n)} \leqslant K \qquad (m, n = 1, 2, \ldots),$$

and let A be the least upper bound of the numbers $A_m^{(n)}$. Then A is the sum of (1). In fact, given any $\varepsilon > 0$, there is a partial sum $A_{m_0}^{(n_0)}$ such that

$$A_{m_0}^{(n_0)} > A - \varepsilon,$$

by the very definition of A. Choosing $m > m_0, n > n_0$, we have

$$A_m^{(n)} > A - \varepsilon,$$

a fortiori, since $A_m^{(n)}$ obviously increases with both m and n. Hence

$$|A_m^{(n)} - A| < \varepsilon$$

if $m, n > N = \max\{m_0, n_0\}$, since no partial sum exceeds A. But then

$$A = \lim_{\substack{m \to \infty \\ n \to \infty}} A_m^{(n)},$$

as asserted. ∎

Next we consider *arbitrary* double series, i.e., double series whose terms can have values of either sign. Just as in the case of ordinary series, we need not consider the case where all the terms of the series are negative or the case where there are only a finite number of positive or negative terms, since these cases immediately reduce to the case of positive series (cf. *Ruds.*, p. 63). Thus we will assume that the double series (1) has both infinitely many positive terms and infinitely many negative terms.

The following definition and theorem are the exact analogues of those for the case of ordinary series (*Ruds.*, pp. 63, 65):

DEFINITION 2. *A double series* (1) *is said to be* **absolutely convergent** *if the series*

$$\sum_{j,k=1}^{\infty} |a_j^{(k)}| \tag{1*}$$

made up of the absolute values of its terms is convergent. A double series which is convergent but not absolutely convergent is said to be **conditionally convergent.**

Remark. Thus (1) is absolutely convergent if and only if the double series corresponding to the infinite rectangular matrix

$$\begin{array}{l} |a_1^{(1)}|\ |a_2^{(1)}| \cdots |a_j^{(1)}| \cdots \\ |a_1^{(2)}|\ |a_2^{(2)}| \cdots |a_j^{(2)}| \cdots \\ \quad .\quad .\quad \cdots\quad .\quad \cdots \\ |a_1^{(k)}|\ |a_2^{(k)}| \cdots |a_j^{(k)}| \cdots \\ \quad .\quad .\quad \cdots\quad .\quad \cdots \end{array}$$

is convergent. Similarly, the interated series (3) and (3′) are absolutely convergent if and only if the iterated series corresponding to this "absolute value matrix" are convergent (cf. Definition 2, p. 33).

THEOREM 3. *If the series* (1*) *converges, then so does* (1).

Proof. Let

$$a_j^{(k)} = p_j^{(k)} - q_j^{(k)},$$

where

$$p_j^{(k)} = \frac{|a_j^{(k)}| + a_j^{(k)}}{2}, \quad q_j^{(k)} = \frac{|a_j^{(k)}| - a_j^{(k)}}{2}.$$

Since

$$p_j^{(k)} \leqslant |a_j^{(k)}|, \quad q_j^{(k)} \leqslant |a_j^{(k)}|,$$

the convergence of (1*) implies that of the double series

$$\sum_{j,\,k=1}^{\infty} p_j^{(k)} = P, \quad \sum_{j,\,k=1}^{\infty} q_j^{(k)} = Q$$

(see Problem 5). But then the series

$$\sum_{j,\,k=1}^{\infty} a_j^{(k)} = \sum_{j,\,k=1}^{\infty} (p_j^{(k)} - q_j^{(k)})$$

also converges (see Problem 3), and in fact has the sum $A = P - Q$. ∎

Finally we prove a result analogous to Theorems 1 and 2 of the preceding section:

THEOREM 4. *Given a double series* (1), *let*

$$\sum_{r=1}^{\infty} u_r \tag{6}$$

be an ordinary series consisting of the same terms written in any order whatsoever. Suppose either of the series (1) *and* (6) *is absolutely convergent. Then so is the other series, and both series have the same sum.*

Proof. Suppose (1) is absolutely convergent, so that (1*) is convergent with sum A^* (say). Then, just as in the proof of Theorem 2, p. 33, it is easy to see that

$$U_r^* \leqslant A^* \qquad (r = 1, 2, \ldots), \tag{7}$$

where

$$U_r^* = |u_1| + |u_2| + \cdots + |u_r|$$

is the rth partial sum of the series

$$\sum_{r=1}^{\infty} |u_r|. \tag{6*}$$

But (7) implies the absolute convergence of (6).

Conversely, suppose (6) is absolutely convergent, so that (6*) is convergent with sum U^*. Then, given any partial sum

$$A_m^{*(n)} = \sum_{j=1}^{m} \sum_{k=1}^{n} |a_j^{(k)}| \tag{8}$$

of the double series (1*), there is an integer r so large that all the terms in the right-hand side of (8) are among the first r terms of (6). It follows that

$$A_m^{*(n)} \leqslant U^* \qquad (m, n = 1, 2, \ldots),$$

which in turn implies the absolute convergence of (1), or, for that matter, of either of the iterated series (3) and (3′).

Finally, to calculate the sum U of the series (6), we use its absolute convergence to suitably rearrange its terms. Suppose

we enumerate the terms of (6) "by squares," amalgamating the groups of terms by which consecutive squares differ from each other in the array

$$\begin{array}{|c|cc|c|c|}
\hline
a_1^{(1)} & a_2^{(1)} & & a_3^{(1)} & \cdots \\
\hline
a_1^{(2)} & a_2^{(2)} & & a_3^{(2)} & \cdots \\
\hline
a_1^{(3)} & a_2^{(3)} & & a_3^{(3)} & \cdots \\
\hline
\cdot & \cdot & & \cdot & \cdots \\
\end{array}$$

Then

$$U = a_1^{(1)} + (a_1^{(2)} + a_2^{(2)} + a_2^{(1)}) + (a_1^{(3)} + a_2^{(3)} + a_3^{(3)} + a_3^{(2)} + a_3^{(1)}) + \cdots,$$

and hence

$$U = \lim_{n \to \infty} A_n^{(n)} = A,$$

where A is the sum of the double series (1). ∎

COROLLARY. *The double series* (1), *the ordinary series* (6) *consisting of the same terms written in any order, and the iterated series* (3) *and* (3′) *all converge and have the same sum, provided at least one of the four series is absolutely convergent.*[3]

Proof. Combine Theorem 4 with Theorems 1–3 of the preceding section. ∎

PROBLEMS

1. Given two convergent series

$$\sum_{j=1}^{\infty} a_j = a_1 + a_2 + \cdots + a_j + \cdots \tag{9}$$

3. The word "absolutely" is unnecessary if all four series are positive.

and

$$\sum_{k=1}^{\infty} b_k = b_1 + b_2 + \cdots + b_k + \cdots, \tag{10}$$

with sums A and B, respectively, consider the double series

$$\sum_{j,\,k=1}^{\infty} c_j^{(k)} = \sum_{j,\,k=1}^{\infty} a_j b_k. \tag{11}$$

Prove that (11) converges and has sum AB.

Hint. In terms of partial sums,

$$C_m^{(n)} = A_m B_n,$$

and hence,

$$C = \lim_{\substack{m\to\infty \\ n\to\infty}} A_m B_n = AB.$$

Comment. Thus, if we define the product of the series (9) and (10) as the *double* series (11), the sum of (11) always equals the product of the sums of (9) and (10). The niceties of Secs. 4–6 arise only when the product of (9) and (10) is taken to be an *ordinary* series.

2. Suppose every term of a convergent double series with sum A is multiplied by the same constant c. Prove that the resulting series converges and has sum cA.

Comment. This problem and the next two generalize familiar properties of ordinary series (*Ruds.*, Theorems 3–5, pp. 7–8).

3. Prove that two convergent double series can be added (or subtracted) term by term.

4. Prove that if the double series

$$\sum_{j,\,k=1}^{\infty} a_j^{(k)}$$

converges, then

$$\lim_{\substack{j\to\infty \\ k\to\infty}} a_j^{(k)} = 0.$$

Hint. Note that

$$a_j^{(k)} = A_j^{(k)} - A_{j-1}^{(k)} - A_j^{(k-1)} + A_{j-1}^{(k-1)}.$$

5. State and prove the analogue for positive double series of the comparison test for ordinary positive series (*Ruds.*, Theorem 1, p. 12).

9. Examples

We now give a number of example illustrating the theory of the preceding two sections.

Example 1. Consider the infinite rectangular matrix

$$\begin{array}{cccccc}
x & -x^2 & x^2 & -x^3 & x^3 & \cdots \\
x(1-x) & -x^2(1-x^2) & x^2(1-x^2) & -x^3(1-x^3) & x^3(1-x^3) & \cdots \\
x(1-x)^2 & -x^2(1-x^2)^2 & x^2(1-x^2)^2 & -x^3(1-x^3)^2 & x^3(1-x^3)^2 & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdots
\end{array}$$

Here the "row series" are absolutely convergent, with sums x, $x(1-x)$, $x(1-x)^2$, ..., and the series made up of these sums is also absolutely convergent, with sum 1. However, the other iterated series does not converge, since the "column series" have sums alternately equal to $+1$ and -1. This does not contradict Theorem 3, p. 34, since neither iterated series is *absolutely* convergent (why not?).

Example 2. Suppose q ranges over all positive integers of the form m^n, where m and n are both positive integers greater than 1, taking each such value just once. Prove that

$$G = \sum_q \frac{1}{q-1} = 1.$$

Solution. If m ranges over all positive integers (>1) *which are*

not powers, then

$$
\begin{aligned}
G &= \sum_m \frac{1}{m^2-1} + \sum_m \frac{1}{m^3-1} + \cdots \\
&= \sum_m \left\{ \frac{1}{m^2-1} + \frac{1}{m^3-1} + \cdots \right\} \\
&= \sum_m \left\{ \left(\frac{1}{m^2} + \frac{1}{m^4} + \frac{1}{m^6} + \cdots \right) \right. \\
&\quad \left. + \left(\frac{1}{m^3} + \frac{1}{m^6} + \frac{1}{m^9} + \cdots \right) + \cdots \right\} \\
&= \sum_m \left\{ \left(\frac{1}{m^2} + \frac{1}{m^3} + \cdots \right) + \left(\frac{1}{m^4} + \frac{1}{m^6} + \cdots \right) \right. \\
&\quad \left. + \left(\frac{1}{m^6} + \frac{1}{m^9} + \cdots \right) + \cdots \right\} \\
&= \sum_m \left\{ \left(\frac{1}{m(m-1)} + \frac{1}{m^2(m^2-1)} + \frac{1}{m^3(m^3-1)} + \cdots \right) \right\}
\end{aligned}
$$

(justify the various rearrangements). It follows that

$$G = \sum_{n=2}^{\infty} \frac{1}{n(n-1)},$$

where n now ranges over *all* positive integers starting from 2. Therefore

$$G = \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right) = 1.$$

Example 3. Consider the matrix with general term

$$a_j^{(k)} = \frac{(k-1)!}{j(j+1)\cdots(j+k)} = \frac{(j-1)!}{k(k+1)\cdots(k+j)}$$

(where $0! = 1$). Setting $\alpha = 0$, $p = k$ in the formula

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)\cdots(\alpha+n+p)} \\
&\qquad = \frac{1}{p(\alpha+p)\cdots(\alpha+p)} \qquad (1)
\end{aligned}
$$

(*Ruds.*, Prob. 2, p. 4), we can easily sum the terms of the kth row:

$$\sum_{j=1}^{\infty} a_j^{(k)} = \frac{(k-1)!}{k \cdot k!} = \frac{1}{k^2}.$$

Hence the sum of one iterated series is

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_j^{(k)} = \sum_{k=1}^{\infty} \frac{1}{k^2}. \tag{2}$$

Because of the symmetry of $a_j^{(k)}$ with respect to j and k, the other iterated series is identical with the first, and nothing new can be deduced by equating the sums of the two iterated series.

Suppose we now modify the matrix as follows: We retain the first $k-1$ terms in the kth row, but replace the kth term by the sum of all the terms of the kth row starting from the kth, dropping the remaining terms altogether. This gives the new matrix

$$\begin{array}{|llllllll}
r_1 & & & & & & & \\
a_1^{(2)} & r_2 & & & & & & \\
a_1^{(3)} & a_2^{(3)} & r_3 & & & & & \\
\cdot & \cdot & \cdot & & & & & \\
a_1^{(k)} & a_2^{(k)} & a_3^{(k)} & \cdots & a_{k-1}^{(k)} & r_k & & \\
a_1^{(k+1)} & a_2^{(k+1)} & a_3^{(k+1)} & \cdots & a_{k-1}^{(k+1)} & a_k^{(k+1)} & r_{k+1} & \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots
\end{array}$$

with the same "row sums" as the original matrix. Hence the sum of the first iterated series has the same value (2) as before. To sum the matrix "by columns," we first calculate

$$\begin{aligned}
r_k = \sum_{j=k}^{\infty} a_j^{(k)} &= \sum_{j=k}^{\infty} \frac{(k-1)!}{j\,(j+1)\cdots(j+k)} \\
&= \sum_{n=1}^{\infty} \frac{(k-1)!}{(k-1+n)\cdots(2k-1+n)} \\
&= \frac{(k-1)!}{k^2\,(k+1)\cdots(2k-1)},
\end{aligned} \tag{3}$$

using formula (1) with $\alpha = k - 1$, $p = k$. The sum of the remaining terms of the kth column equals

$$\sum_{i=k+1}^{\infty} a_k^{(i)} = \sum_{i=k+1}^{\infty} \frac{(k-1)!}{i(i+1)\cdots(i+k)}$$

$$= \sum_{n=1}^{\infty} \frac{(k-1)!}{(k+n)(k+n+1)\cdots(2k+n)}$$

$$= \frac{(k-1)!}{k(k+1)\cdots 2k}, \tag{4}$$

where we set $\alpha = p = k$ in (1). It follows from (3) and (4) that the sum of all the terms in the kth column is just

$$\frac{(k-1)!}{k(k+1)\cdots(2k-1)}\left(\frac{1}{k} + \frac{1}{2k}\right)$$

$$= 3\,\frac{(k-1)!}{k(k+1)\cdots(2k-1)\,2k} = 3\,\frac{[(k-1)!]^2}{(2k)!}.$$

Using Theorem 3, p. 34 to equate the sums of the two iterated series, we arrive at the interesting formula

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 3\sum_{k=1}^{\infty} \frac{[(k-1)!]^2}{(2k)!}. \tag{5}$$

The series on the right converges very rapidly, thereby facilitating the approximate calculation of the sum of the important series on the left.[4]

Example 4. Investigate the convergence of the double series

$$\sum_{j,k=1}^{\infty} \frac{1}{(j+k)^{\sigma}} \qquad (\sigma > 0). \tag{6}$$

4. An argument which will not be given here shows that this sum equals $\pi^2/6$. See G.M. Fichtenholz, *Functional Series* (in the Pocket Mathematical Library), Sec. 7, Problem 7b.

Solution. Arrange the terms of (6) as an ordinary series, combining terms on each diagonal of the corresponding matrix. Since the terms on each diagonal are equal, this gives

$$\sum_{n=2}^{\infty} (n-1)\frac{1}{n^{\sigma}}. \tag{7}$$

But clearly

$$\frac{1}{2}n \leqslant n-1 < n,$$

and hence, dividing by n^{σ}, we get

$$\frac{1}{2}\frac{1}{n^{\sigma-1}} \leqslant (n-1)\frac{1}{n^{\sigma}} < \frac{1}{n^{\sigma-1}}.$$

It follows by the comparison test that (7) converges if $\sigma > 2$ and diverges if $\sigma < 2$. Therefore, by Theorem, 4 p. 40, the same is true of the double series (6).

PROBLEMS

1. By applying Theorem 3, p. 34 to the positive matrix

$$\begin{array}{|cccccc}
\frac{1}{1\cdot 2} & \frac{1}{2\cdot 3} & \frac{1}{3\cdot 4} & \frac{1}{4\cdot 5} & \cdots \\
 & \frac{1}{2\cdot 3} & \frac{1}{3\cdot 4} & \frac{1}{4\cdot 5} & \cdots \\
 & & \frac{1}{3\cdot 4} & \frac{1}{4\cdot 5} & \cdots \\
 & & & \frac{1}{4\cdot 5} & \cdots \\
 & & & & \cdots \\
 & & & &
\end{array}$$

(the missing elements can be replaced by zeros), prove that the harmonic series

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} + \cdots \tag{8}$$

must diverge.

Hint. If (8) had a finite sum s, then one iterated sum would equal s, while the other would equal $s - 1$, contrary to the theorem.

2. Prove that

$$\sum_{m=2}^{\infty} \sum_{k=2}^{\infty} \frac{1}{m^k} = 1$$

(the same power can occur more than once in the left-hand side).

3. If $|x| < 1$, the *Lambert series*

$$\varphi(x) = \sum_{n=1}^{\infty} a_n \frac{x^n}{1 - x^n} \tag{9}$$

converges for precisely the same values of x as the power series

$$f(x) = \sum_{n=1}^{\infty} a_n x^n \tag{10}$$

(*Ruds.*, Prob. 3, p. 86). Let $R > 0$ be the radius of convergence of the series (10) and suppose $|x| < R$ as well as $|x| < 1$. Prove that

$$\varphi(x) = \sum_{n=1}^{\infty} \alpha_n x^n,$$

where

$$\alpha_n = \sum_{k|n} a_k$$

and $k|n$ means that the sum is restricted to those positive integers which are divisors of n.

Hint. Noting that

$$\frac{x^n}{1 - x^n} = x^n + x^{2n} + \cdots + x^{kn} + \cdots \qquad (|x| < 1),$$

consider the matrix

$$
\begin{array}{llllllllll}
a_1x & a_1x^2 & a_1x^3 & a_1x^4 & a_1x^5 & a_1x^6 & a_1x^7 & a_1x^8 & a_1x^9 & a_1x^{10}\cdots \\
a_2x^2 & & a_2x^4 & & a_2x^6 & & a_2x^8 & & a_2x^{10}\cdots & \\
a_3x^3 & & & & a_3x^6 & & & a_3x^9 & \cdots & \\
a_4x^4 & & & & & & a_4x^8 & & \cdots & \\
a_5x^5 & & & & & & & & a_5x^{10}\cdots & \\
a_6x^6 & & & & & & & & \cdots & \\
a_7x^7 & & & & & & & & \cdots & \\
a_8x^8 & & & & & & & & \cdots & \\
a_9x^9 & & & & & & & & \cdots & \\
a_{10}x^{10}\cdots & & & & & & & & & \\
\cdot & & & & & & & & \cdots &
\end{array}
\tag{11}
$$

where identical powers of x appear in the same column and empty spaces can be replaced by zeros. Now equate the corresponding iterated series (why is this justified?).

4. Prove that

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} \tau(n)\, x^n,$$

$$\sum_{n=1}^{\infty} \frac{nx^n}{1-x^n} = \sum_{n=1}^{\infty} \sigma(n)\, x^n,$$

where $\tau(n)$ is the number of divisors of n and $\sigma(n)$ is the sum of the divisors of n.

Hint. Use the result of the preceding problem.

Comment. It is easily verified that $R = 1$ in both cases, so that we need only assume that $|x| < 1$.

5. Prove that

$$\varphi(x) = \sum_{n=1}^{\infty} f(x^n), \tag{12}$$

where the functions φ and f are the same as in Problem 3.

Hint. Arrange the elements of the matrix (11) without "gaps," in the form

$$\begin{array}{|llll} a_1x & a_1x^2 & a_1x^3 & a_1x^4 \cdots \\ a_2x^2 & a_2x^4 & a_2x^6 & a_2x^8 \cdots \\ a_3x^3 & a_3x^6 & a_3x^9 & a_3x^{12} \cdots \\ a_4x^4 & a_4x^8 & a_4x^{12} & a_4x^{16} \cdots \\ \cdot & \cdot & \cdot & \cdot \cdots \end{array}$$

6. Prove that

$$\sum_{n=1}^{\infty} \frac{(ax)^n}{1 - x^n} = \sum_{n=1}^{\infty} \frac{ax^n}{1 - ax^n} \qquad (|a| < 1, |x| < 1).$$

Hint. Use (12), choosing $a_n = a^n$.

7. Given two power series

$$f(x) = \sum_{m=1}^{\infty} a_m x^m, \quad g(x) = \sum_{n=1}^{\infty} b_n x^n,$$

prove that

$$\sum_{m=1}^{\infty} a_m g(x^m) = \sum_{n=1}^{\infty} b_n f(x^n) \qquad (|x| < 1),$$

where x is such that both series converge absolutely.

Hint. Consider the matrix with elements $a_m b_n x^{mn}$, noting that

$$|a_m b_n x^{mn}| \leqslant |a_m x^m|\, |b_n x^n|,$$

since $mn \geqslant m + n$ $(m > 1, n > 1)$, from which it follows that the corresponding double series is absolutely convergent.

Comment. The choice $b_n = 1$ $(n = 1, 2, \ldots)$ gives (12), since then

$$g(x) = \frac{x}{1 - x}.$$

8. Prove that the "power series in two variables"

$$\sum_{j,k=0}^{\infty} x^j y^k \tag{13}$$

is absolutely convergent if $|x| < 1$, $|y| < 1$ and divergent otherwise.

Hint. Use Cauchy's theorem (p. 14), noting that (13) is the product of the two ordinary power series

$$\sum_{j=0}^{\infty} x^j, \quad \sum_{k=0}^{\infty} y^k.$$

Comment. The study of power series in two variables is pursued in the next section.

9. Prove that the double series

$$\sum_{j,k=1}^{\infty} \frac{1}{j^\alpha k^\beta} \qquad (\alpha > 0, \beta > 0)$$

is convergent if $\alpha > 1$, $\beta > 1$ and divergent otherwise.

10. Consider the double series

$$\sum_{j,k=1}^{\infty} a_j^{(k)} \equiv \sum_{j,k=1}^{\infty} \frac{1}{(Aj^2 + 2Bjk + Ck^2)^\varrho} \qquad (\varrho > 0), \tag{14}$$

where the quadratic form $Ax^2 + 2Bxy + Cy^2$ is positive definite, so that $\Delta = AC - B^2 > 0$, as well as $A > 0$, $C > 0$. Prove that (14) converges if $\varrho > 1$ and diverges otherwise, thereby generalizing Example 4.

Hint. If M is the largest of the numbers $|A|$, $|B|$ and $|C|$, then

$$a_j^{(k)} \geqslant \frac{1}{M^\varrho} \frac{1}{(j + k)^{2\varrho}}.$$

Moreover

$$a_j^{(k)} = \frac{1}{C}\left[(AC - B^2)\,j^2 + (Bj + Ck)^2\right] \geqslant \frac{\Delta}{C}\,j^2,$$

and hence

$$a_j^{(k)} \leqslant \left(\frac{C}{\Delta}\right)^{\varrho} \frac{1}{j^{2\varrho}}, \quad a_j^{(k)} \leqslant \left(\frac{A}{\Delta}\right)^{\varrho} \frac{1}{k^{2\varrho}},$$

which implies

$$a_j^{(k)} \leqslant \left(\frac{\sqrt{AC}}{\Delta}\right)^{\varrho} \frac{1}{j^{\varrho}k^{\varrho}}.$$

11. Give an example showing that in Theorem 1, p. 36, one cannot drop the condition that every row series be convergent.

Hint. The double series corresponding to the matrix

$$\begin{vmatrix}
1 & -1 & 1 & -1 & \cdots \\
-1 & 1 & -1 & 1 & \cdots \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \cdots \\
-\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \cdots \\
\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \cdots \\
-\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdots
\end{vmatrix}$$

is convergent, with sum zero, but its row series diverge.

12. Sum the following double series:

a) $\sum_{m,n=2}^{\infty} \frac{1}{(p+n)^m}$ $(p > -1)$; b) $\sum_{m=2,n=1}^{\infty} \frac{1}{(2n)^m}$;

c) $\sum_{m,n=1}^{\infty} \frac{1}{(4n-1)^{2m+1}}$; d) $\sum_{m,n=1}^{\infty} \frac{1}{(4n-1)^{2m}}$;

e) $\sum_{m,n=1}^{\infty} \frac{1}{(4n-2)^{2m}}$.

Hint. Resort to iterated series, first summing over m. Use the expansions

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots,$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

(*Ruds.*, pp. 115, 117).

Ans.

a) $\dfrac{1}{p+1}$; b) $\ln 2$; c) $\dfrac{\pi}{8} - \dfrac{1}{2}\ln 2$; d) $\dfrac{1}{4}\ln 2$; e) $\dfrac{\pi}{8}$.

13. Consider the function of two variables

$$\varphi(x, z) = e^{x/2\,(z - z^{-1})} \quad (z \neq 0).$$

Prove that[5]

$$\varphi(x, z) = \sum_{n=-\infty}^{\infty} J_n(x)\, z^n, \tag{15}$$

where

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+n)!} \left(\frac{x}{2}\right)^{2k+n}$$

if $n \geqslant 0$ and

$$J_n(x) = \sum_{k=-n}^{\infty} \frac{(-1)^k}{k!\,(k+n)!} \left(\frac{x}{2}\right)^{2k+n}$$

if $n < 0$. Show that

$$J_{-n}(x) = (-1)^n J_n(x).$$

Hint. Multiply the absolutely convergent series

$$e^{(x/2)z} = \sum_{j=0}^{\infty} \left(\frac{x}{2}\right)^j \frac{z^j}{j!}, \quad e^{-(x/2)z^{-1}} = \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k \frac{(-1)^k z^{-k}}{k!}$$

(*Ruds.*, p. 113) to get the absolutely convergent double series

$$\varphi(x, z) = \sum_{j,k=0}^{\infty} \left(\frac{x}{2}\right)^{j+k} \frac{(-1)^k}{j!\,k!}\, z^{j-k}.$$

5. $\sum_{n=-\infty}^{\infty} a_n$ is shorthand for the sum $\sum_{n=0}^{\infty} a_n + \sum_{n=1}^{\infty} a_{-n}$.

Comment. The function $J_n(x)$ is called the *Bessel function of order n*, and plays an important role in mathematical physics, celestial mechanics, etc. Because of the expansion (15), the function $\varphi(x, z)$ is called the "generating function" of the Bessel functions.

10. Power Series in Two Variables

By a *power series in two variables x and y* is meant a double series of the form

$$\sum_{j,k=0}^{\infty} a_{jk} x^j y^k, \tag{1}$$

involving nonnegative integral powers of the variables x and y multiplied by numerical coefficients a_{jk}. Just as in the case of ordinary power series in a single variable (*Ruds.*, Sec. 11), we now consider the problem of finding the "region of convergence" of (1), i.e., the set $\mathcal{M}$ of all points $M = (x, y)$ for which (1) converges.

LEMMA. *If the power series* (1) *converges at a point* $\bar{M} = (\bar{x}, \bar{y})$ *whose coordinates are both nonzero, then it converges absolutely at every point* $M = (x, y)$ *satisfying the inequalities* $|x| < |\bar{x}|$, $|y| < |\bar{y}|$, *i.e., in the whole open rectangle with center at the origin and vertex at* $\bar{M}$.

Proof. The proof is virtually the same as in the single-variable case (*Ruds.*, p. 68). Suppose the series

$$\sum_{j,k=0}^{\infty} a_{jk} \bar{x}^j \bar{y}^k$$

converges. Then its general term approaches zero (see Problem 4, p. 42), and hence is bounded, i.e.,

$$|a_{jk} \bar{x}^j \bar{y}^k| \leqslant C \qquad (j, k = 0, 1, 2, \ldots)$$

for some constant C. Therefore

$$|a_{jk} x^j y^k| = |a_{jk} \bar{x}^j \bar{y}^k| \left| \frac{x}{\bar{x}} \right|^j \left| \frac{y}{\bar{y}} \right|^k \leqslant C \left| \frac{x}{\bar{x}} \right|^j \left| \frac{y}{\bar{y}} \right|^k.$$

But the expression on the right is the general term of a convergent double series (see Problem 8, p. 51), provided that $|x| < |\bar{x}|$, $|y| < |\bar{y}|$. It follows by the comparison test (see Problem 5, p. 43) that (1) is absolutely convergent and hence convergent, by Theorem 3, p. 39. ∎

Remark 1. The only power series in two variables to be considered here are those for which there exists a point $\bar{M}$ satisfying the conditions of the lemma.

Remark 2. It is clear from the lemma that we need only study the behavior of the series (1) in the first quadrant of the xy-plane, since if (1) converges at a point $M = (x, y)$, it must also converge at the points obtained by reflecting M in the x and y-axes as well as in the origin.

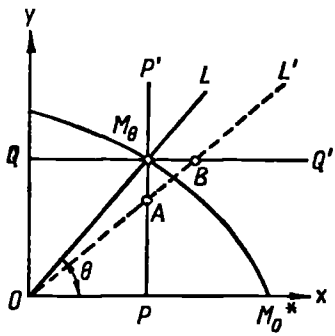


Figure 1

Consider a ray OL in the first quadrant, making angle θ with the positive x-axis (see Figure 1). Then, by the exact analogue of the argument in the single-variable case (*Ruds.*, p. 69), there is a positive number $R(\theta)$, possibly infinite, such that the series (1) converges absolutely for every point M on OL satisfying the condition[6]

$$\overline{OM} < R(\theta)$$

and diverges for every point M on OL satisfying the condition

$$\overline{OM} > R(\theta).$$

6. The distance from O to M is denoted by $\overline{OM}$.

Note that if $R(\theta) = +\infty$ for even a single ray, then, by the lemma, the series (1) converges (absolutely) in the whole xy-plane, i.e., the "region of convergence" reduces to the whole xy-plane.

Excluding this case of an "everywhere convergent" series, we now assume that $R(\theta)$ is finite for all θ. Let M_θ be the point of the ray OL such that

$$\overline{OM}_\theta = R(\theta).$$

Then the "boundary point" M_θ separates the points of the ray for which (1) converges (absolutely) from the points for which (1) diverges. At the point M_θ itself, the series (1) may or may not converge, depending on the particular series under consideration. Drawing the vertical line PP' and the horizontal line QQ' through M_θ, as shown in the figure, we can assert (using the lemma) that the series certainly converges inside the rectangle $OPM_\theta Q$ and diverges inside the (right) angular sector $Q'M_\theta P'$. Therefore on a new ray OL', corresponding to another angle θ', the series must converge at every point of OA and diverge at every point of BL'. Hence the boundary point M_θ, must lie between A and B on OL'. This makes it clear (why?) that $R(\theta)$ varies *continuously* as θ varies from 0 to $\pi/2$. In other words, the point M_θ describes a continuous "boundary curve" as θ varies from 0 to $\pi/2$. Since the abscissa x_θ of the point M_θ is nondecreasing as $\theta \to 0$ while the ordinate y_θ of M_θ is nonincreasing, x_θ and y_θ must both approach limits as $\theta \to 0$. Hence $R(\theta)$ must also approach a limit as $\theta \to 0$. Suppose this limit

$$R_0 = \lim_{\theta \to 0} R(\theta), \tag{2}$$

is finite. Then, as $\theta \to 0$, M_θ approaches a limiting point $M_0^* = (R_0, 0)$ on the x-axis.[7] If (2) is infinite, the boundary curve must

7. The limiting point M_0^* need not coincide with the boundary point M_0 on the x-axis itself. In fact, M_0 may lie to the right of M_0^* (or even lie at infinity). This possibility does not contradict the lemma, which applies only to points *off* the coordinate axes.

have an asymptote parallel to the x-axis, which may coincide with the x-axis itself. The same considerations apply to the case where $\theta \to \pi/2$, with the roles of the x and y-axes interchanged.

The above construction gives the boundary curve in the first quadrant. To complete the construction, we now reflect this portion of the boundary curve in both axes and in the origin. This gives the full boundary curve, determining the "region of convergence" $\mathcal{M}$. In fact, the series (1) converges in the part of the plane lying inside the boundary curve and diverges in the part of the plane lying outside the curve.[8] On the boundary curve itself, the series may or may not converge, depending on circumstances.

PROBLEMS

1. Sketch the region of convergence for the series

$$\sum_{j,k=0}^{\infty} x^j y^k \tag{3}$$

(already considered in Problem 8, p. 51).

Ans. The square shown in Figure 2.

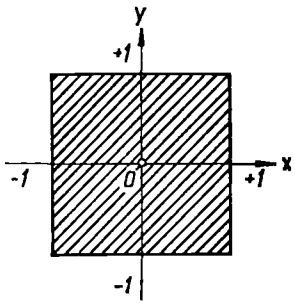


Figure 2

8. Except possibly on the parts of the coordinate axes lying outside the curve, where it may turn out that the series converge (see footnote 7).

2. What is the sum of the series (3) in its region of convergence?

Ans. $\dfrac{1}{1-x}\,\dfrac{1}{1-y}$.

3. Find the region of convergence of the series

$$\sum_{j,k=1}^{\infty} x^j y^k,$$

differing from (3) in that the summation indices begin with 1 rather than 0.

Ans. The same square as in Problem 1, together with the coordinate axes themselves.

Comment. In this case, even though the boundary point M_θ approaches the point $M_0^* = (1, 0)$ as $\theta \to 0$, the series converges on the whole x-axis (cf. footnotes 7 and 8).

4. Find the region of convergence of the series

$$\sum_{j,k=0}^{\infty} \frac{x^j y^k}{j!\,k!}.$$

Ans. The whole xy-plane.

5. Find the region of convergence of the series

$$\sum_{j,k=0}^{\infty} \frac{(j+k)!}{j!\,k!}\, x^j y^k. \tag{4}$$

Hint. A necessary and sufficient condition for the absolute convergence of the series (4), i.e., for the convergence of the series

$$\sum_{j,k=0}^{\infty} \frac{(j+k)!}{j!\,k!}\, |x|^j\, |y|^k, \tag{4'}$$

is the convergence of the series

$$\sum_{n=0}^{\infty} (|x| + |y|)^n = \frac{1}{1 - |x| - |y|}$$

obtained by summing (4') "by diagonals." This leads to the condition $|x| + |y| < 1$.

Ans. The square shown in Figure 3.

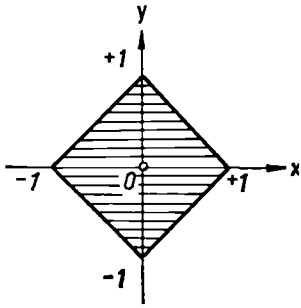


Figure 3

6. Find the region of convergence of the series

$$\sum_{j \geqslant k} x^j y^k = 1 + x + x^2 + \cdots + xy + x^2y + x^3y + \cdots$$
$$+ x^2y^2 + x^3y^2 + x^4y^2 + \cdots . \qquad (5)$$

Hint. A necessary and sufficient condition for the absolute convergence of the series (5), i.e., for the convergence of the series

$$\sum_{j \geqslant k} |x|^j |y|^k, \qquad (5')$$

is the convergence of the series

$$(1 + |x| + |x|^2 + \cdots)(1 + |xy| + |xy|^2 + \cdots)$$
$$= \frac{1}{1 - |x|} \frac{1}{1 - |xy|}$$

obtained by summing (5') "by rows." This leads to the conditions $|x| < 1$, $|xy| < 1$.

Ans. The region shown in Figure 4, bounded in part by the equilateral hyperbolas $xy = \pm 1$.

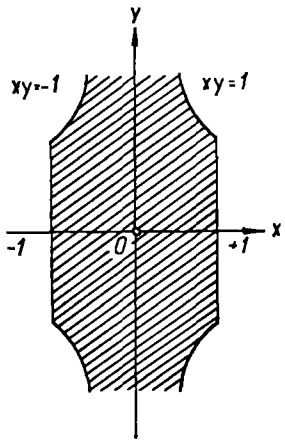


Figure 4

7. Generalizing double series, discuss *multiple series* of the form[9]

$$\sum_{j,k,\ldots,l=1}^{\infty} a_{jk\cdots l} \tag{6}$$

and *power series in several variables*, of the form

$$\sum_{j,k,\ldots,=0}^{\infty} a_{jk\ldots l} x^j y^k \cdots z^l .$$

How should the sum of a series like (6) be defined?

9. In discussing iterated and double series, we favored the notation $a_j^{(k)}$ over the simpler notation a_{jk}, merely to emphasize the distinction between rows and columns in the infinite matrices figuring in Secs. 7–9.

CHAPTER 3

Computations Involving Series

11. General Remarks

We now consider the problem of using infinite series to make *approximate computations*. Suppose we want to calculate a number A which is known to have a series expansion of the form

$$A = a_1 + a_2 + \cdots + a_n + \cdots, \tag{1}$$

where the terms $a_1, a_2, \ldots$ are *readily computable*.[1] Then we have the approximation

$$A \approx A_n = a_1 + a_2 + \cdots + a_n,$$

where A_n is the nth partial sum of the series (1). The *error* committed in making this approximation is obviously just the sum of the omitted terms, i.e., the remainder

$$\alpha_n = a_{n+1} + a_{n+2} + \cdots$$

(after n terms). Clearly, α_n can be made arbitrarily small for sufficiently large n. Our job is to find simple estimates for α_n. We will then be in a position to determine the value of n for which A_n approximates A to within any given accuracy.

If (1) is an alternating series whose successive terms decrease in absolute value, i.e., if (1) is a series of the *Leibniz type*, then the remainder α_n has the same sign as its first term and a smaller absolute value than its first term (*Ruds.*, p. 74). Thus, in this

1. In fact, the terms $a_1, a_2, \ldots$ are usually rational numbers.

case, we already have at our disposal a technique for estimating α_n, which leaves nothing to be desired from the standpoint of simplicity. The situation is more complicated for a positive series. To estimate the error α_n, we look for another, *easily summable* positive series whose terms are larger than the corresponding terms of α_n. For example, in the case of the series

$$\sum_{m=1}^{\infty} \frac{1}{m^2},$$

we have the estimate[2]

$$\alpha_n = \sum_{m=n+1}^{\infty} \frac{1}{m^2} < \sum_{m=n+1}^{\infty} \frac{1}{m(m-1)}$$
$$= \sum_{m=n+1}^{\infty} \left(\frac{1}{m-1} - \frac{1}{m} \right) = \frac{1}{n}.$$

Remark. One is ordinarily interested in decimal approximations to the number A, although the terms $a_1, a_2, \ldots$ rarely turn out to be terminating decimals. Thus the "round-off error" committed in writing $a_1, a_2, \ldots$ in decimal form must be taken into account, as well as the "truncation error" α_n.

Even when the series (1) has simple terms and a remainder which is easily estimated, it often fails to be a *practical* way of calculating A, due to its having too slow a rate of convergence. In other words, the partial sums $A_1, A_2, \ldots$ may approach their limit A too slowly as $n \to \infty$. For example, the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots,$$
$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \tag{2}$$

converge to $\ln 2$ and $\pi/4$, respectively (cf. Problem 12, p. 52). However, to calculate these numbers to within 10^{-5}, we need 100,000 terms in the first case and 50,000 terms in the second

2. The same estimate has already been found in *Ruds.*, formula (6), p. 45, in connection with the integral test.

case! Obviously, such calculations are feasible only with the help of a high-speed calculating machine. As will be shown in the next section, it is not very hard to find series expansions of π and ln 2 which converge much more rapidly than (2).

PROBLEM

Let α_n be the remainder after n terms of the series

$$1 + \sum_{m=1}^{\infty} \frac{1}{m!}.$$

Prove that

$$\alpha_n < \frac{1}{n!\, n}.$$

Hint. Note that

$$\sum_{m=n+1}^{\infty} \frac{1}{m!} = \frac{1}{n!} \sum_{m=n+1}^{\infty} \frac{1}{(n+1)\cdots m} < \frac{1}{n!} \sum_{m=n+1}^{\infty} \frac{1}{(n+1)^{m-n}}.$$

12. Examples

The following examples illustrate the technique of using series to make numerical computations:

Example 1. Find a rapidly convergent series suitable for calculating π.

Solution. Setting $x = 1/\sqrt{3}$ in the expansion

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad (-1 \leqslant x \leqslant 1)$$

(*Ruds.*, p. 115), we obtain

$$\frac{\pi}{6} = \arctan \frac{1}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} \left(1 - \frac{1}{3}\frac{1}{3} + \frac{1}{5}\frac{1}{3^2} - \frac{1}{7}\frac{1}{3^3} + \cdots\right),$$

which is much more suitable for calculating π than the slowly convergent series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

To get a series which is even more suitable for calculating π, let

$$\alpha = \arctan \frac{1}{5}.$$

Then

$$\tan \alpha = \frac{1}{5}, \quad \tan 2\alpha = \frac{\frac{2}{5}}{1 - \frac{1}{25}} = \frac{5}{12},$$

$$\tan 4\alpha = \frac{\frac{10}{12}}{1 - \frac{25}{144}} = \frac{120}{119}.$$

Since $\tan 4\alpha$ is close to 1, it is clear that the angle 4α is close to $\pi/4$. Setting

$$\beta = 4\alpha - \frac{\pi}{4},$$

we have

$$\tan \beta = \frac{\frac{120}{119} - 1}{1 + \frac{120}{119}} = \frac{1}{239},$$

so that

$$\beta = \arctan \frac{1}{239}.$$

It follows that

$$\begin{aligned} \pi &= 16\alpha - 4\beta \\ &= 16\left(\frac{1}{5} - \frac{1}{3}\frac{1}{5^3} + \frac{1}{5}\frac{1}{5^5} - \frac{1}{7}\frac{1}{5^7} + \frac{1}{9}\frac{1}{5^9}\right. \\ &\quad \left. - \frac{1}{11}\frac{1}{5^{11}} + \cdots\right) - 4\left(\frac{1}{239} - \frac{1}{3}\frac{1}{239^3} + \cdots\right). \end{aligned} \tag{1}$$

This series is very rapidly convergent (see Problem 2).

Example 2. Find a rapidly convergent series suitable for calculating ln 2.

Solution. Setting

$$x = \frac{1}{2n+1}$$

in the expansion

$$\ln \frac{1+x}{1-x} = 2x\left(1 + \frac{x^2}{3} + \frac{x^4}{5} + \cdots\right) \quad (|x| < 1)$$

(*Ruds.*, p. 117), we get

$$\ln \frac{n+1}{n} = \frac{2}{2n+1}\left[1 + \frac{1}{3}\frac{1}{(2n+1)^2} + \frac{1}{5}\frac{1}{(2n+1)^4} + \cdots\right] \tag{2}$$

(cf. *Ruds.*, p. 124). For $n = 1$, this gives the rapidly convergent series

$$\ln 2 = \frac{2}{3}\left(1 + \frac{1}{3}\frac{1}{9} + \frac{1}{5}\frac{1}{9^2} + \cdots\right) \tag{3}$$

(see Problem 3).

Example 3. One way of calculating roots is to use a table of logarithms. Another way is to use the binomial series

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots \quad (|x| < 1) \tag{4}$$

(*Ruds.*, p. 120). Let a be an approximate value of the root $\sqrt[k]{A}$, where a may be an overestimate or an underestimate. To improve this approximation, suppose, say, that

$$\frac{A}{a^k} = 1 + x,$$

where $|x|$ is a small proper fraction. Then

$$\sqrt[k]{A} = a\sqrt[k]{\frac{A}{a^k}} = a(1 + x)^{1/k}, \tag{5}$$

and we can use (4) with $m = 1/k$. Alternatively, it is often more convenient to start from the formula

$$\frac{a^k}{A} = 1 + x',$$

where $|x'|$ is again a small proper fraction. This time

$$\sqrt[k]{A} = \frac{a}{\sqrt[k]{\frac{a^k}{A}}} = a(1 + x')^{-1/k}, \tag{5'}$$

and we can use (4) with $m = -1/k$.

Thus, for example, to sharpen the approximation

$$\sqrt{2} \approx 1.4,$$

we write

$$\sqrt{2} = 1.4\sqrt{\frac{2}{1.96}} = 1.4\sqrt{1 + \frac{0.04}{1.96}} = 1.4\left(1 + \frac{1}{49}\right)^{1/2}$$

or

$$\sqrt{2} = \frac{1.4}{\sqrt{\frac{1.96}{2}}} = \frac{1.4}{\sqrt{1 - \frac{0.04}{2}}} = 1.4\left(1 - \frac{1}{50}\right)^{-1/2}.$$

To keep the calculations simple, we naturally prefer the second formula, which leads to the rapidly convergent expansion

$$\sqrt{2} = 1.4\left(1 + \frac{1}{2}\frac{1}{50} + \frac{3}{8}\frac{1}{50^2} + \frac{5}{16}\frac{1}{50^3} + \frac{35}{128}\frac{1}{50^4} + \frac{63}{256}\frac{1}{50^5} + \cdots\right) \tag{6}$$

(see Problem 8).

PROBLEMS

1. Prove that

$$\frac{\pi}{4} = \left(\frac{1}{2} - \frac{1}{3}\frac{1}{2^3} + \frac{1}{5}\frac{1}{2^5} - \cdots\right) + \left(\frac{1}{3} - \frac{1}{3}\frac{1}{3^3} + \frac{1}{5}\frac{1}{3^5} - \cdots\right).$$

Hint. Choose $x = \frac{1}{2}$, $y = \frac{1}{3}$ in the formula

$$\arctan x + \arctan y = \arctan \frac{x + y}{1 - xy},$$

valid for

$$|\arctan x + \arctan y| < \frac{\pi}{2}$$

(explain this condition).

2. Use formula (1) to calculate π to seven decimal places, making sure to take round-off error into account.

Hint. It is enough to compute the terms already written out in (1).

Ans. $\pi = 3.1415926\ldots$

3. Use formula (3) to calculate ln 2 to nine decimal places.

Hint. Nine terms of the series suffice. In fact, if Δ is the error due to dropping all terms from the tenth on, then

$$\Delta = \frac{2}{3}\left(\frac{1}{19}\frac{1}{9^9} + \frac{1}{21}\frac{1}{9^{10}} + \cdots\right) < \frac{2}{3 \cdot 19 \cdot 9^9}\left(1 + \frac{1}{9} + \frac{1}{9^2} + \cdots\right) = \frac{1}{12 \cdot 19 \cdot 9^8} < \frac{2}{10^{10}}.$$

Ans. $\ln 2 = 0.693147180\ldots$

4. Show that

$$\ln 5 = 2 \ln 2 + \frac{2}{9}\left(1 + \frac{1}{3}\,\frac{1}{81} + \frac{1}{5}\,\frac{1}{81^2} + \cdots\right). \qquad (7)$$

Hint. Set $n = 4$ in formula (2).

5. Using (3) and (7), calculate

$$M = \frac{1}{\ln 10}$$

to nine decimal places.

Ans. $M = 0.434294481\ldots$

Comment. Note that[3]

$$\log n = M \ln n,$$

so that M is the ratio of the common logarithm of any number to its natural logarithm. Thus, going over to common logarithms in (2) and (3), we get

$$\log \frac{n+1}{n} = \frac{2M}{2n+1} \times \left[1 + \frac{1}{3}\,\frac{1}{(2n+1)^2} + \frac{1}{5}\,\frac{1}{(2n+1)^4} + \cdots\right] \qquad (2')$$

and

$$\log 2 = \frac{2M}{3}\left(1 + \frac{1}{3}\,\frac{1}{9} + \frac{1}{5}\,\frac{1}{9^2} + \cdots\right). \qquad (3')$$

6. Use formula (3′) to find log 3 and log 7.

Hint. Choose $n = 80 = 2^3 \cdot 10$, noting that $n + 1 = 81 = 3^4$. Then

$$4 \log 3 - 3 \log 2 - 1 = \frac{2M}{161}\left(1 + \frac{1}{3}\,\frac{1}{25{,}921} + \frac{1}{5}\,\frac{1}{25{,}921^2} + \cdots\right),$$

3. As always, $\log n = \log_{10} n$, $\ln n = \log_e n$.

from which log 3 can easily be found (log 2 is already known). Next choose $n = 2400 = 3 \cdot 2^3 \cdot 10^2$, noting that $n + 1 = 2401 = 7^4$. Then

$$4 \log 7 - 3 \log 2 - \log 3 - 2$$
$$= \frac{2M}{4801}\left(1 + \frac{1}{3}\,\frac{1}{23{,}049{,}601} + \frac{1}{5}\,\frac{1}{23{,}049{,}601^2} + \cdots\right).$$

Comment. This technique can be used to find the common logarithms of the prime numbers to within any desired accuracy. Common logarithms of composite numbers can then be found by adding integral multiples of common multiples of prime numbers. For example,

$$\log 120 = 3 \log 2 + \log 3 + \log 5.$$

7. Prove that the error Δ of the approximation

$$\log (n + 1) - \log n \approx \frac{2M}{2n + 1} \qquad (10^3 \leqslant n \leqslant 10^4) \qquad (8)$$

is less than $\frac{1}{2} \times 10^{-10}$.

Hint. Let Δ be the error of the approximation (8). Then

$$\Delta = \frac{2M}{2n + 1}\left[\frac{1}{3}\,\frac{1}{(2n + 1)^2} + \frac{1}{5}\,\frac{1}{(2n + 1)^4} + \cdots\right]$$
$$< \frac{2M}{3(2n + 1)^3}\left[1 + \frac{1}{(2n + 1)^2} + \frac{1}{(2n + 1)^4} + \cdots\right]$$
$$= \frac{2M}{3(2n + 1)\,2n\,(2n + 2)} < \frac{2M}{24n^3}.$$

Comment. Even if all the separate errors have the same sign, the use of (8) to calculate logarithms of the numbers 1000 to 10,000 step by step would lead to a cumulative error of less than

$$\frac{10^4}{2 \cdot 10^{10}} = \frac{1}{2} \times 10^{-6}.$$

However, it is easy to avoid such accumulation of errors by using (2′) to calculate a number of "control logarithms." In this way, we can achieve much higher accuracy while at the same time retaining the automatic character of formula (8), a feature which is very valuable in compiling extensive tables.

8. Use formula (6) to calculate $\sqrt{2}$ to ten decimal places.

Hint. It is enough to compute the terms already written out in (6). If Δ is the error committed in dropping all subsequent terms, then

$$\Delta < 1.4 \frac{231}{1024 \cdot 50^6}\left(1 + \frac{1}{50} + \frac{1}{50^2} + \cdots\right)$$

$$= 1.4 \frac{231}{1024 \cdot 50^5 \cdot 49} < \frac{2.1}{10^{11}}.$$

Ans. $\sqrt{2} = 1.4142135623\ldots$

Comment. To get an even more accurate value of $\sqrt{2}$, we can start from the formula

$$\sqrt{2} = 1.41\left(1 - \frac{119}{20{,}000}\right)^{-1/2}.$$

9. Verify that

$$\sqrt{3} = 1.73\left(1 - \frac{71}{30{,}000}\right)^{-1/2}, \quad \sqrt{11} = \frac{10}{3}\left(1 - \frac{1}{100}\right)^{1/2},$$

$$\sqrt[3]{2} = \frac{5}{4}\left(1 + \frac{3}{125}\right)^{1/3}, \qquad \sqrt[3]{3} = \frac{10}{7}\left(1 + \frac{29}{1000}\right)^{1/3}.$$

Use these formulas to get accurate approximations to the roots in question.

13. Euler's Transformation

In using a series for approximate calculations, it is sometimes useful to subject it to a preliminary *transformation*. By this we mean the use of some rule or other to replace the given series by

another series with the same sum. Of course, making such a transformation is appropriate only when the new series converges more rapidly than the old series or is more suitable for calculations.

We now derive the formula for a classical transformation due to Euler. Let

$$S(x) = \sum_{k=0}^{\infty} (-1)^k a_k x^k$$

$$= a_0 - a_1 x + a_2 x^2 - \cdots + (-1)^k a_k x^k + \cdots \qquad (x > 0) \qquad (1)$$

be a *convergent* series, where for convenience we write, the kth term in the form $(-1)^k a_k x^k$ without assuming that all the $a_k > 0$. For the sequence $\{a_k\}$ we introduce the successive differences

$$\Delta a_k = a_{k+1} - a_k, \Delta^2 a_k = \Delta a_{k+1} - \Delta a_k = a_{k+2} - 2a_{k+1} + a_k,$$

and, in general,

$$\Delta^p a_k = \Delta^{p-1} a_{k+1} - \Delta^{p-1} a_k$$

$$= a_{k+p} - C_1^p a_{k+p-1} + C_2^p a_{k+p-2} - \cdots + (-1)^p a_k, \qquad (2)$$

where C_j^p is the binomial coefficient

$$C_j^p = \frac{p!}{j!\,(p-j)!}.$$

We then write the series (1) in the form

$$S(x) = \frac{a_0}{1+x} - \frac{a_1 x - a_0 x}{1+x}$$

$$+ \frac{a_2 x^2 - a_1 x^2}{1+x} - \frac{a_3 x^3 - a_2 x^3}{1+x} + \cdots. \qquad (3)$$

This is permissible since the $(k + 1)$th partial sum of the new series differs from the kth sum of (1) only by the term

$$\frac{1}{1 + x}(-1)^{k+1} a_{k+1} x^{k+1},$$

which approaches zero as $k \to \infty$ because of the convergence of the original series (*Ruds.*, Theorem 5, p. 8).

We now use the differences to simplify (3), obtaining

$$S(x) = \frac{1}{1 + x}(a_0 - \Delta a_0 \cdot x + \Delta a_1 \cdot x^2 - \Delta a_2 \cdot x^3 + \cdots).$$

Retaining the first term, we write the remaining series

$$-\frac{x}{1 + x}(\Delta a_0 - \Delta a_1 \cdot x + \Delta a_2 \cdot x^2 - \cdots)$$

like $S(x)$ itself in the form

$$-\frac{x}{1 + x}\frac{1}{1 + x}(\Delta a_0 - \Delta^2 a_0 \cdot x + \Delta^2 a_1 \cdot x^2 - \cdots),$$

so that, splitting off the first term again, we get

$$S(x) = \frac{a_0}{1 + x} - \frac{\Delta a_0}{(1 + x)^2} x + \frac{x^2}{(1 + x)^2}(\Delta^2 a_0 - \Delta^2 a_1 \cdot x + \cdots).$$

Continuing in this way, we obtain

$$S(x) = \frac{a_0}{1 + x} - \frac{\Delta a_0}{(1 + x)^2} x + \frac{\Delta^2 a_0}{(1 + x)^3} - \cdots + (-1)^{p-1} \frac{\Delta^{p-1} a_0}{(1 + x)^p} x^{p-1} + R_p(x) \qquad (4)$$

after p steps, where

$$R_p(x) = (-1)^p \frac{x^p}{(1+x)^p} (\Delta^p a_0 - \Delta^p a_1 \cdot x + \Delta^p a_2 \cdot x^2 - \cdots)$$

$$= (-1)^p \frac{x^p}{(1+x)^p} \sum_{k=0}^{\infty} (-1)^k \Delta^p a_k x^k.$$

To prove that $R_p(x) \to 0$ as $p \to \infty$, we replace the pth difference $\Delta^p a_k$ by its expression (2) and reverse the order of summation, obtaining

$$R_p(x) = \frac{1}{(1+x)^p} \sum_{k=0}^{\infty} (-1)^{k+p} x^{k+p} \sum_{i=0}^{p} (-1)^i C_i^p a_{k+p-i}$$

$$= \frac{1}{(1+x)^p} \sum_{i=0}^{p} C_i^p x^i \sum_{k=0}^{\infty} (-1)^{k+p-i} a_{k+p-i} x^{k+p-i}.$$

Denoting the remainder of the original series (1) by

$$r_n(x) = \sum_{k=0}^{\infty} (-1)^{k+n} a_{k+n} x^{k+n} \qquad (n = 0, 1, 2, \ldots),$$

we can finally write $R_p(x)$ in the form

$$R_p(x) = \frac{\sum_{i=0}^{p} C_i^p x^i r_{p-i}(x)}{(1+x)^p} = \frac{\sum_{i=0}^{p} C_i^p x^{p-i} r_i(x)}{(1+x)^p},$$

and hence $R_p(x) \to 0$ as $p \to \infty$, by Problem 3, p. 25, since $r_n(x) \to 0$ as $n \to \infty$.

Taking the limit as $p \to \infty$ in (4), we get

$$S(x) = \frac{1}{1+x} \left\{ a_0 - \Delta a_0 \frac{x}{1+x} + \Delta^2 a_0 \left(\frac{x}{1+x}\right)^2 - \cdots \right.$$

$$\left. + (-1)^p \Delta^p a_0 \left(\frac{x}{1+x}\right)^p + \cdots \right\}. \tag{5}$$

Comparing (1) and (5), we finally obtain *Euler's transformation*

$$\sum_{k=0}^{\infty}(-1)^k a_k x^k = \frac{1}{1+x}\sum_{p=0}^{\infty}(-1)^p \Delta^p a_0 \left(\frac{x}{1+x}\right)^p. \tag{6}$$

Equation (6) is most frequently used with $x = 1$, and it then transforms one numerical series into another:

$$\sum_{k=0}^{\infty}(-1)^k a_k = \sum_{p=0}^{\infty}\frac{(-1)^p \Delta^p a_0}{2^{p+1}}. \tag{7}$$

Example 1. Let

$$a_k = \frac{1}{z+k},$$

where z is any constant different from $0, -1, -2, \ldots$ Then the series

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{z+k}$$

is a series of the Leibniz type if we drop a sufficiently large number of initial terms, and hence converges. The successive differences $\Delta a_k, \Delta^2 a_k, \ldots$ are easily calculated. In fact, using mathematical induction, we find that

$$\Delta^p a_k = (-1)^p \frac{p!}{(z+k)(z+k+1)\cdots(z+k+p)},$$

and, in particular,

$$\Delta^p a_0 = (-1)^p \frac{p!}{z(z+1)\cdots(z+p)}. \tag{8}$$

It follows from (7) and (8) that

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{z+k} = \sum_{p=0}^{\infty}\frac{1}{2^{p+1}}\frac{p!}{z(z+1)\cdots(z+p)}. \tag{9}$$

Setting $z = 1$ in (9) leads to the following transformation of the familiar series for $\ln 2$:

$$\ln 2 = \sum_{m=1}^{\infty}(-1)^{m-1}\frac{1}{m} = \sum_{n=1}^{\infty}\frac{1}{2^n n}.$$

The second series is obviously more suitable than the first for numerical calculations. In fact, to get an accuracy of 0.01 we need 99 terms of the first sequence and only 5 terms of the second!

Example 2. Let

$$a_k = \frac{1}{z + 2k},$$

where z is a constant different from $0, -2, -4, \ldots$ Representing a_k in the form

$$a_k = \frac{1}{2} \frac{1}{\frac{z}{2} + k},$$

we can use formula (8) to write

$$\Delta^p a_0 = (-1)^p \frac{1}{2} \frac{p!}{\frac{z}{2}\left(\frac{z}{2} + 1\right) \cdots \left(\frac{z}{2} + p\right)}$$

$$= (-1)^p \frac{1}{2} \frac{2^{p+1} p!}{z(z + 2) \cdots (z + 2p)}. \tag{10}$$

Then Euler's transformation (7) takes the form

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{z + 2k} = \frac{1}{2} \sum_{p=0}^{\infty} \frac{p!}{z(z + 2) \cdots (z + 2p)}. \tag{11}$$

Setting $z = 1$ in (11) leads to the following transformation of the familiar series for $\pi/4$ (*Ruds.*, p. 115):

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k + 1} = \frac{1}{2} \sum_{p=0}^{\infty} \frac{p!}{(2p + 1)(2p - 1) \cdots 3 \cdot 1}.$$

Example 3. In calculations using a transformation of a series, it is often convenient to first calculate some terms of the series *directly* and afterwards subject only the *remainder* of the series

to the transformation. As an illustration, suppose we calculate π by using the series

$$\pi = 2\left(1 + \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \cdots + \frac{1 \cdot 2 \cdots p}{3 \cdot 5 \cdots (2p + 1)} + \cdots\right), \tag{12}$$

derived in the preceding example. Since the ratio of the last written term to the preceding term is

$$\frac{p}{2p + 1} < \frac{1}{2},$$

the discarded remainder of the series is always less than the preceding calculated term. For example, we get π to six decimal places by calculating 21 terms of the series (12), since

$$2\,\frac{1 \cdot 2 \cdot 3 \cdots 20}{1 \cdot 3 \cdot 5 \cdots 41} = 0.00000037\ldots < 0.0000005.$$

However, if we calculate the first 7 terms, say, of the original series directly and transform only the remainder after 7 terms, we get

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13}\right) - 2\left(\frac{1}{15} + \frac{1}{15 \cdot 17} + \frac{1 \cdot 2}{15 \cdot 17 \cdot 19} + \cdots + \frac{1 \cdot 2 \cdots p}{15 \cdot 17 \cdots (15 + 2p)} + \cdots\right).$$

The eighth term of the transformed series is now within the required limits, since

$$2\,\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}{15 \cdot 17 \cdots 29} = 0.0000002\ldots$$

Hence to get the same accuracy as before, we must now only calculate 8 more terms besides the 7 retained terms, i.e. only 15 terms in all as opposed to the previous 21 terms!

PROBLEMS

1. Prove that

$$\arctan x = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} x^{2k+1}$$

$$= \frac{x}{1+x^2} \sum_{p=0}^{\infty} \frac{2p\,(2p-2)\cdots 4\cdot 2}{(2p+1)\,(2p-1)\cdots 3\cdot 1} \left(\frac{x^2}{1+x^2}\right)^p$$

$$(0 \leqslant x \leqslant 1)$$

(cf. *Ruds.*, p. 115).

Hint. Choose $a_k = 1/(2k+1)$ in (6), using (10) with $z = 1$ to calculate $\Delta^p a_k$.

2. Give examples showing that subjecting a convergent series to Euler's transformation does not always improve its rate of convergence.

Comment. In comparing the rate of convergence of two series

$$\sum_{k=0}^{\infty} c_k, \qquad \sum_{k=0}^{\infty} c_k'$$

with terms of arbitrary sign, we start from the behavior of the ratio of the corresponding remainders γ_n and γ_n', i.e., if $|\gamma_n/\gamma_n'| \to 0$ as $n \to \infty$, we say that the first series converges *more rapidly* than the second and the second *more slowly* than the first (cf. Ruds., Definition 1, p. 53).

Ans. The series

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{2^k}$$

is transformed into the more rapidly convergent series

$$\sum_{p=0}^{\infty} \frac{1}{2} \frac{1}{4^p},$$

while the series

$$\sum_{k=0}^{\infty} \frac{1}{2^k}$$

is transformed into the more slowly convergent series

$$\sum_{p=0}^{\infty} \frac{1}{2} \left(\frac{3}{4}\right)^p.$$

14. The Transformations of Kummer and Markov

We have just seen how Euler's transformation, based on a well-defined rule, leads to a unique transformed series, which, however, is not always suitable for computational purposes (see Problem 2 above). We now describe a method of transforming series due to Kummer which has a high degree of built-in arbitrariness (requiring much ingenuity on the part of the calculator), but which in return is more specifically directed to the goal of simplifying the given calculation.

Thus let

$$A^{(1)} + A^{(2)} + \cdots + A^{(k)} + \cdots \tag{1}$$

be a convergent series, and suppose we want to calculate the sum of (1) to a given accuracy. Clearly $A^{(k)} \to 0$ as $k \to \infty$. Suppose we find another series

$$a_1^{(1)} + a_1^{(2)} + \cdots + a_1^{(k)} + \cdots \tag{2}$$

converging to an *easily calculable* finite sum A_1 such that the general terms $A^{(k)}$ and $a_1^{(k)}$ of the series (1) and (2) are *equivalent*, indicated by writing $A^{(k)} \sim a_1^{(k)}$ (as $k \to \infty$), in the sense that

$$\lim_{k \to \infty} \frac{\alpha_1^{(k)}}{A^{(k)}} = 0, \tag{3}$$

where

$$\alpha_1^{(k)} = A^{(k)} - a_1^{(k)}. \tag{4}$$

Formula (3) is usually written more concisely as

$$\alpha_1^{(k)} = o(A^{(k)}). \tag{5}$$

Note that (5) implies

$$\alpha_1^{(k)} = o(a_1^{(k)})$$

as well, since

$$\lim_{k\to\infty} \frac{\alpha_1^{(k)}}{a_1^{(k)}} = \lim_{k\to\infty} \frac{\alpha_1^{(k)}}{A^{(k)} - \alpha_1^{(k)}} = \lim_{k\to\infty} \frac{\dfrac{\alpha_1^{(k)}}{A^{(k)}}}{1 - \dfrac{\alpha_1^{(k)}}{A^{(k)}}} = 0.$$

It follows from (4) that

$$\sum_{k=1}^{\infty} A^{(k)} = \sum_{k=1}^{\infty} a_1^{(k)} + \sum_{k=1}^{\infty} \alpha_1^{(k)} = A_1 + \sum_{k=1}^{\infty} \alpha_1^{(k)}. \tag{6}$$

Thus we have reduced the calculation of the sum of the original series (1) to that of the sum of a transformed series whose terms certainly approach zero more rapidly than $A^{(k)}$, because of (5).

Example. To calculate the sum of the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2},$$

we recall that

$$\sum_{k=1}^{\infty} \frac{1}{k\,(k+1)} = 1$$

(*Ruds.*, Prob. 2, p. 4). Since

$$\lim_{k\to\infty} \frac{\dfrac{1}{k^2} - \dfrac{1}{k\,(k+1)}}{\dfrac{1}{k^2}} = \lim_{k\to\infty} \frac{\dfrac{1}{k^2\,(k+1)}}{\dfrac{1}{k^2}} = 0,$$

it follows from (6) that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \sum_{k=1}^{\infty} \frac{1}{k^2 (k+1)}, \tag{7}$$

where the transformed series is clearly more suitable for calculation than the original series.

Repeating the same process, suppose we can find still another series

$$a_2^{(1)} + a_2^{(2)} + \cdots + a_2^{(k)} + \cdots$$

converging to an easily calculable finite sum A_2 such that $a_2^{(k)}$ is equivalent to $\alpha_1^{(k)}$ in the indicated sense. Then

$$\sum_{k=1}^{\infty} A^{(k)} = A_1 + A_2 + \sum_{k=1}^{\infty} \alpha_2^{(k)},$$

so that the calculation of the sum of the original series (1) reduces to that of the sum of a series whose terms

$$\alpha_2^{(k)} = \alpha_1^{(k)} - a_2^{(k)} = o(\alpha_1^{(k)})$$

converge to zero more rapidly than $\alpha_1^{(k)}$. Similarly, repeating the process p times, we arrive at the formula

$$\sum_{k=1}^{\infty} A^{(k)} = A_1 + A_2 + \cdots + A_p + \sum_{k=1}^{\infty} \alpha_p^{(k)}, \tag{8}$$

where

$$A_j = \sum_{k=1}^{\infty} a_j^{(k)} \qquad (j = 1, 2, \ldots, p)$$

are known sums of series which are successively "split off" from the original series, so that the problem reduces to calculating the sum of the series in the right-hand side of (8).

Next we discuss another method, due to Markov, for transforming a given convergent series

$$\sum_{k=1}^{\infty} A^{(k)} = A. \tag{9}$$

Like Kummer's method, this method also has much built-in arbitrariness.

Suppose every term of (9) can be represented as the sum of a convergent series

$$A^{(k)} = \sum_{j=1}^{\infty} a_j^{(k)} \qquad (k = 1, 2, \ldots), \tag{10}$$

and consider the infinite rectangular matrix

$$\begin{array}{c|l} A^{(1)} & a_1^{(1)}\ a_2^{(1)}\ a_3^{(1)} \ldots a_j^{(1)} \ldots \\ A^{(2)} & a_1^{(2)}\ a_2^{(2)}\ a_3^{(2)} \ldots a_j^{(2)} \ldots \\ A^{(3)} & a_1^{(3)}\ a_2^{(3)}\ a_3^{(3)} \ldots a_j^{(3)} \ldots \\ \cdot & \cdot\ \cdot\ \cdot \ldots \cdot \ldots \\ A^{(k)} & a_1^{(k)}\ a_2^{(k)}\ a_3^{(k)} \ldots a_j^{(k)} \ldots \\ \cdot & \cdot\ \cdot\ \cdot \ldots \cdot \ldots \end{array} \tag{11}$$

made up of the terms of the series (10). Then the number A is just the sum of the iterated series

$$A = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_j^{(k)}$$

corresponding to this matrix. Assuming that every column of the matrix gives a convergent series

$$\sum_{k=1}^{\infty} a_j^{(k)} = A_j,$$

Markov established a necessary and sufficient condition for the series

$$\sum_{j=1}^{\infty} A_j$$

to converge to the same sum A. Correspondingly, *Markov's transformation* consists in replacing the original iterated series

(summed over rows) by the other iterated series (summed over columns):

$$A = \sum_{k=1}^{\infty} A^{(k)} = \sum_{j=1}^{\infty} A_j.$$

A sufficient condition for applying Markov's transformation is given by Theorem 3, p. 34, say. However, the theorem proved by Markov is much more general, and does not even assume the absolute convergence of the series involved.

PROBLEMS

1. Prove that $\alpha_k \sim \beta_k$ (as $k \to \infty$) if and only if

$$\lim_{k\to\infty} \frac{\alpha_k}{\beta_k} = 1.$$

2. Show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + 2! \sum_{k=1}^{\infty} \frac{1}{k^2 (k+1)(k+2)},$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + 3! \sum_{k=1}^{\infty} \frac{1}{k^2 (k+1)(k+2)(k+3)},$$

and more generally, that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \cdots + \frac{1}{p^2} + p! \sum_{k=1}^{\infty} \frac{1}{k^2 (k+1) \cdots (k+p)}. \tag{12}$$

Hint. Starting from (7), use (8) and the formula

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)\cdots(k+p)} = \frac{1}{p \cdot p!}$$

(*Ruds.*, Prob. 2, p. 4).

Comment. Thus the calculation of the sum of the slowly converging series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

is reduced to the calculation of the sum of p of its terms and the sum of a transformed series which *converges rapidly* (even for moderate values of p).

3. Show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 3\left(\frac{1}{2} + \frac{1}{2\cdot 2^2}\frac{1}{3} + \frac{1}{3\cdot 2^3}\frac{2!}{5\cdot 3\cdot 1} + \cdots + \frac{1}{p\cdot 2^p}\frac{(p-1)!}{(2p-1)(2p-3)\cdots 3\cdot 1}\right) + \frac{(p!)^3}{2^p(2p-1)(2p-3)\cdots 3\cdot 1} \times \times \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)^2\cdots(k+p)^2}. \tag{13}$$

Hint. Since

$$\frac{1}{2p-1}\left[\frac{k+y}{k^2(k+1)^2\cdots(k+p-1)^2} - \frac{k+1+y}{(k+1)^2(k+2)^2\cdots(k+p)^2}\right] = \frac{1}{2p-1}\left[\frac{(2p-1)k^2 + p(p+2y)k + yp^2}{k^2(k+1)^2\cdots(k+p-1)^2(k+p)^2}\right] \sim \frac{1}{k^2(k+1)^2\cdots(k+p-1)^2}$$

as $k \to \infty$, where y is temporarily undetermined, it follows from (6) that

$$S_p \equiv \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)^2\cdots(k+p-1)^2} = \frac{1}{2p-1}\frac{1+y}{(p!)^2} + \sum_{k=1}^{\infty} \frac{\left[2p - \frac{p}{2p-1}(p+2y)\right]k + \left[p^2 - \frac{yp^2}{2p-1}\right]}{k^2(k+1)^2\cdots(k+p)^2}.$$

Now choose

$$y = \frac{3p}{2} - 1,$$

thereby causing the term containing k to vanish in the numerator of the series on the right. With this choice of p,

$$S_p = \frac{3p}{2\,(2p-1)\,(p!)^2} + \frac{p^3}{2\,(2p-1)}\,S_{p+1}.$$

Therefore

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = S_1 = \frac{3}{2} + \frac{1}{2}\,S_2,$$

$$\frac{1}{2}\,S_2 = \frac{3}{2\cdot 2^2}\,\frac{1}{3} + \frac{(2!)^3}{2^2\cdot 3\cdot 1}\,S_3,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\frac{[(p-1)!]^3}{2^{p-1}\,(2p-3)\,(2p-5)\cdots 3\cdot 1}\,S_p$$

$$= \frac{3}{p\cdot 2^p}\,\frac{(p-1)!}{(2p-1)\,(2p-3)\cdots 3\cdot 1}$$

$$+ \frac{(p!)^3}{2^p\,(2p-1)\,(2p-3)\cdots 3\cdot 1}\,S_{p+1},$$

which together imply (13).

4. Suppose we choose $p = 5$ in (13) and also retain 5 terms of the transformed series. What is the accuracy of the resulting calculation of the sum of the original series?

5. Interpret Example 3, p. 44 as an instance of Markov's transformation, discussing the results of using the two representations

$$\frac{1}{k^2} = a_1^{(k)} + a_2^{(k)} + \cdots + a_{k-1}^{(k)} + r_k,$$

$$\frac{1}{k^2} = \sum_{j=1}^{\infty} a_j^{(k)},$$

where

$$a_j^{(k)} = \frac{(k-1)!}{j(j+1)\cdots(j+k)}.$$

Comment. As already noted on p. 45, Markov's transformation gives nothing new if we use the second representation.

6. Show the connection between Kummer's transformation and Markov's transformation.

Hint. Take the limit as $p \to \infty$ in formula (8), assuming that

$$\lim_{p\to\infty} \sum_{k=1}^{\infty} a_p^{(k)} = 0. \tag{14}$$

7. Verify that (14) holds in the case of the expansion (13), thereby proving that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 3\left(\frac{1}{2} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3}\frac{2!}{5\cdot 3\cdot 1} + \cdots + \frac{1}{p\cdot 2^p}\frac{(p-1)!}{(2p-1)(2p-3)\cdots 3\cdot 1} + \cdots\right)$$

$$= 3\sum_{p=1}^{\infty} \frac{1}{p\cdot 2^p}\frac{(p-1)!}{(2p-1)(2p-3)\cdots 3\cdot 1}. \tag{15}$$

Hint. The sum in the last term in the right-hand side of (13) does not exceed

$$\frac{1}{(p!)^2}\sum_{k=1}^{\infty}\frac{1}{k^2},$$

and hence the whole term does not exceed the quantity

$$\frac{p!}{2^p(2p-1)(2p-3)\cdots 3\cdot 1}\sum_{k=1}^{\infty}\frac{1}{k^2},$$

which obviously approaches zero as $p \to \infty$.

Comment. Note that (15) is equivalent to formula (5), p. 46.

8. Give an example showing that taking the limit (14) does not always lead to a useful result.

Hint. Taking the limit as $p \to \infty$ in (12) merely gives the identity

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{p=1}^{\infty} \frac{1}{p^2}.$$

Comment. Thus Markov's method is a very general technique offering the calculator many possibilities but requiring much ingenuity on his part.

CHAPTER 4

Summation of Divergent Series

15. Introduction

So far, we have defined the sum A of a given numerical series

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots + a_n + \cdots \tag{1}$$

as the *limit*

$$A = \lim_{n \to \infty} A_n \tag{2}$$

of its *partial sums*

$$A_n = a_0 + a_1 + a_2 + \cdots + a_n,$$

assuming that this limit exists and is finite or equal to $\pm\infty$. On the other hand, the sum of an "oscillating" divergent series, for which the limit (2) fails to exist, has not been defined, and we have systematically avoided such series. In the second half of the nineteenth century, however, consideration of various situations encountered in mathematical analysis, such as the fact that the product of two convergent series can diverge (see Example 1, p. 27), naturally brought to the fore the problem of trying to *sum divergent series* in some appropriate *new* sense, perforce different from the usual sense associated with formula (2). Certain of these "summation" methods (to be studied in detail below) turned out to be particularly fruitful.

It should be pointed out that divergent series were quite often encountered in mathematical practice prior to the creation by Cauchy of a rigorous theory of infinite series, based in turn on a

rigorous theory of limits. Although the use of divergent series in proofs was disputed, nevertheless attempts were sometimes made to assign numerical values to such series. For example, the number $\frac{1}{2}$ was assigned to the oscillating series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

even in Leibniz's time, a choice which Euler justified by observing that the expansion

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

(which is actually valid only for $|x| < 1$) becomes

$$\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

if x is formally set equal to 1. There was a kernel of truth in this observation, but the whole approach lacked clarity. For example one could equally well deduce the formula

$$\frac{3}{2} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

by setting $x = 1$ in the expansion

$$\frac{1+x}{1+x+x^2} = \frac{1-x^2}{1-x^3} = 1 - x^2 + x^3 - x^5 + x^6 - x^8 + \cdots.$$

The whole problem is posed differently in modern analysis. The starting point of all discussion is some precisely stated definition of the "generalized sum" of a series, which is applicable to a whole class of numerical series rather than being contrived for a particular series of interest at the moment. The legitimacy of this approach cannot be contested. The reader need only recall that even the ordinary concept of the "sum of a series," as simple and natural as it may seem, was introduced on the basis of a tentative definition, subsequently justified only by its expediency! However, regardless of how such a "generalized sum" is defined, it is usually required to satisfy the following two conditions:

1) If the series

$$\sum_{n=1}^{\infty} a_n$$

is assigned the generalized sum A and the series

$$\sum_{n=1}^{\infty} b_n$$

is assigned the generalized sum B, then the series

$$\sum_{n=1}^{\infty} (pa_n + qb_n),$$

where p and q are arbitrary constants, must be assigned the generalized sum $pA + qB$. A method of summation satisfying this condition is said to be *linear*.

2) The new definition of summation must include the ordinary definition as a special case. More exactly, a series which converges to the sum A in the ordinary sense must have a generalized sum, and this sum must also equal A. A method of summation satisfying this condition is said to be *regular*.

Naturally, we are only interested in regular methods of summation which allow us to sum a larger class of series than can be summed by the ordinary method of summation, since only in this case does it actually make sense to talk about "generalized summation."

The next few sections are devoted to two methods of generalized summation of particular importance in the applications.

PROBLEM

Justify writing

$$\frac{m}{n} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots,$$

where m and n are arbitrary positive integers.

Hint. Note that

$$\frac{1 + x + \cdots + x^{m-1}}{1 + x + \cdots + x^{n-1}} = \frac{1 - x^m}{1 - x^n}.$$

16. The Method of Power Series

First we consider the *method of power series*, primarily due to Poisson who applied it to the study of trigonometric series.

DEFINITION. *Given a numerical series*

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots + a_n + \cdots,$$

suppose the associated power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

converges for all $0 < x < 1$ *to a sum function* $f(x)$, *and suppose that*

$$\lim_{x \to 1-} f(x) = A.$$

Then the number A is called the ***generalized sum*** **(in the sense of Poisson)** *of the given series.*

Example 1. As already noted on p. 88, the series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

has the generalized sum $\frac{1}{2}$ in the sense of Poisson, since

$$1 - x + x^2 - x^3 + x^4 - x^5 + \cdots = \frac{1}{1 + x}$$

and

$$\lim_{x \to 1-} \frac{1}{1 + x} = \frac{1}{2}.$$

Example 2. Use the method of power series to sum the divergent series

$$\frac{1}{2} + \sum_{n=1}^{\infty} \cos n\theta \qquad (-\pi \leqslant \theta \leqslant \pi) \tag{1}$$

(see Problem 1).

Solution. Form the power series

$$\frac{1}{2} + \sum_{n=1}^{\infty} x^n \cos n\theta \qquad (0 < x < 1),$$

whose sum is easily seen to be[1]

$$\frac{1}{2} \frac{1 - x^2}{1 - 2x \cos \theta + x^2}. \tag{2}$$

But

$$\lim_{x \to 1-} \frac{1}{2} \frac{1 - x^2}{1 - 2x \cos \theta + x^2} = 0$$

if $\theta \neq 0$. Hence the generalized sum of the series (1) equals 0 if $\theta \neq 0$. If $\theta = 0$, (2) reduces to

$$\frac{1}{2} \frac{1 - x^2}{(1 - x)^2} = \frac{1}{2} \frac{1 + x}{1 + x},$$

which approaches $+\infty$ as $x \to 1-$. Hence (1) has the generalized sum $+\infty$ if $\theta = 0$. Note that (1) obviously has the ordinary sum $+\infty$ if $\theta = 0$.

It is immediately clear that the present method of generalized summation is *linear*. Its *regularity* is a consequence of the follow-

1. In fact, multiplying

$$1 + 2 \sum_{n=1}^{\infty} x^n \cos n\theta$$

by $1 - 2x \cos \theta + x^2$ and noting that

$$2 \cos nx \cos x = \cos (n + 1)x + \cos (n - 1)x,$$

we get $1 - x^2$ after cancelling terms.

ing theorem, proved by Abel in his investigations on the theory of the binomial series. Accordingly, the method of power series will henceforth be called the *Poisson–Abel method* (of summation).[2]

THEOREM 1 **(Abel).** *If the series*

$$\sum_{n=0}^{\infty} a_n$$

converges to A (in the ordinary sense), then the power series

$$\sum_{n=0}^{\infty} a_n x^n \tag{3}$$

converges for $0 < x < 1$ *and its sum approaches A as* $x \to 1-$.

Proof. First we note that the radius of convergence of the series (3) is no less than 1 (*Ruds.*, p. 68), and hence (1) actually converges for $0 < x < 1$. Consider the identity

$$\sum_{n=0}^{\infty} a_n x^n = (1 - x) \sum_{n=0}^{\infty} A_n x^n, \tag{4}$$

where

$$A_n = a_0 + a_1 + \ldots + a_n$$

(see Prob. 3, p. 83 or *Ruds.*, Prob. 1, p. 85). Subtracting (4) term by term from the obvious identity

$$A = (1 - x) \sum_{n=0}^{\infty} A x^n,$$

we get

$$A - \sum_{n=0}^{\infty} a_n x^n = (1 - x) \sum_{n=0}^{\infty} \alpha_n x^n, \tag{5}$$

2. There is no doubt that the general formulation of the method of power series, used by Poisson only in a special case, stems from Abel's theorem. The Poisson–Abel method is often simply called *Abel's method*, although Abel himself regarded the idea of "summing" a divergent series as very much of an oddity.

where $\alpha_n = A - A_n$. Since $\alpha_n \to 0$ as $n \to \infty$, given any $\varepsilon > 0$, we can find an integer $N > 0$ such that $|\alpha_n| < \varepsilon$ for all $n > N$. The right-hand side of (5) is the sum of the two expressions

$$(1 - x) \sum_{n=0}^{\infty} \alpha_n x^n, \qquad (1 - x) \sum_{n=N+1}^{\infty} \alpha_n x^n, \tag{6}$$

the second of which can be estimated at once (independently of x):

$$\left|(1 - x) \sum_{n=N+1}^{\infty} \alpha_n x^n\right| \leqslant (1 - x) \sum_{n=N+1}^{\infty} |\alpha_n| x^n$$

$$< \varepsilon (1 - x) \sum_{n=N+1}^{\infty} x^n < \varepsilon. \tag{7}$$

As for the first of the expressions (6), it approaches zero as $x \to 1-$, and hence

$$\left|(1 - x) \sum_{n=0}^{N} \alpha_n x^n\right| < \varepsilon \tag{8}$$

for x sufficiently near 1. Combining (7) and (8), we finally get

$$\left|A - \sum_{n=1}^{\infty} a_n x^n\right| < 2\varepsilon. \quad \blacksquare$$

We have already seen that even if the series

$$\sum_{n=0}^{\infty} a_n \tag{9}$$

is Poisson–Abel summable with sum A, it need not have a sum in the ordinary sense. In other words, the existence of the limit

$$\lim_{x \to 1-} \sum_{n=0}^{\infty} a_n x^n = A \tag{10}$$

does not in general imply the convergence of (9). The question naturally arises of what extra conditions must be imposed on the terms of (9) to make (10) imply the convergence of (9), so that

(9) has the sum A in the ordinary sense. The first result along these lines is the following theorem due to Tauber:

THEOREM 2 **(Tauber).** *Suppose the series*

$$\sum_{n=0}^{\infty} a_n x^n$$

converges for $0 < x < 1$ *and satisfies* (10). *Moreover, suppose the terms of the series* (9) *are such that*

$$\lim_{n\to\infty} \frac{a_1 + 2a_2 + \cdots + na_n}{n} = 0. \tag{11}$$

Then

$$\sum_{n=0}^{\infty} a_n = A.$$

Proof. The proof will be carried out in two steps. First we assume that[3]

$$\lim_{n\to\infty} na_n = 0, \tag{12}$$

or equivalently

$$a_n = o\left(\frac{1}{n}\right).$$

Thus if

$$\delta_n = \max_{k\geqslant n} |ka_k|,$$

then δ_n decreases monotonically to zero as $n \to \infty$. Given any integer $N > 0$, we have

$$\sum_{n=0}^{N} a_n - A = \sum_{n=0}^{N} a_n (1 - x^n) - \sum_{n=N+1}^{\infty} a_n x^n + \left(\sum_{n=0}^{\infty} a_n x^n - A\right),$$

and hence

$$\left|\sum_{n=0}^{N} a_n - A\right| \leqslant \sum_{n=0}^{N} |na_n| (1 - x) + \sum_{n=N+1}^{\infty} \frac{|na_n|\, x^n}{n} + \left|\sum_{n=0}^{\infty} a_n x^n - A\right|$$

$$\leqslant (1 - x) N\delta_0 + \frac{\delta_{N+1}}{(N+1)(1-x)} + \left|\sum_{n=0}^{\infty} a_n x^n - A\right|,$$

3. Note that (12) implies (11), by Corollary 2, p. 23, but not conversely, so that we are first proving the theorem under a stronger hypothesis than (11).

where we use the inequalities

$$1 - x^n = (1 - x)(1 + x + x^2 + \cdots + x^{n-1}) < n(1 - x),$$

$$\sum_{n=N+1}^{\infty} x^n = \frac{x^{N+1}}{1 - x} < \frac{1}{1 - x},$$

valid for $0 < x < 1$. Choosing $\varepsilon > 0$ arbitrarily small, we set

$$(1 - x) N = \varepsilon,$$

or equivalently

$$x = 1 - \frac{\varepsilon}{N},$$

so that $x \to 1$ as $N \to \infty$. Now let N be so large that $\delta_{N+1} < \varepsilon^2$ and the corresponding value of x is close enough to 1 to make

$$\left| \sum_{n=0}^{\infty} a_n x^n - A \right| < \varepsilon.$$

Then

$$\left| \sum_{n=0}^{N} a_n - A \right| < (2 + \delta_0)\, \varepsilon,$$

which, because of the arbitrariness of N and ε, proves the theorem under the hypothesis (12).

We now reduce the more general case of the theorem to the special case just considered. Let

$$v_0 = 0,$$

$$v_n = a_1 + 2a_2 + \cdots + na_n \qquad (n \geqslant 1),$$

so that

$$a_n = \frac{1}{n}(v_n - v_{n-1}) \qquad (n \geqslant 1)$$

and hence

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} \frac{v_n}{n} x^n - \sum_{n=1}^{\infty} \frac{v_{n-1}}{n} x^n$$

$$= a_0 + (1 - x) \sum_{n=1}^{\infty} \frac{v_n}{n} x^n + \sum_{n=1}^{\infty} \frac{v_n}{n(n+1)} x^{n+1}. \quad (13)$$

This time we have (11) or, equivalently, $v_n/n \to 0$ as $n \to \infty$. It follows that

$$\lim_{x \to 1-} (1 - x) \sum_{n=1}^{\infty} \frac{v_n}{n} x^n = 0. \tag{14}$$

To see this, we write

$$(1 - x) \sum_{n=1}^{\infty} \frac{v_n}{n} x^n = (1 - x) \sum_{n=1}^{N} \frac{v_n}{n} x^n + (1 - x) \sum_{n=N+1}^{\infty} \frac{v_n}{n} x^n,$$

and then choose N so large that all the factors v_n/n in the second expression on the right have absolute values less than some preassigned number $\varepsilon > 0$. Then the second expression is itself less than ε, regardless of x. Moreover, the first expression on the right, which contains only a finite number of terms, can also be made less than ε, by simply choosing x close enough to 1.

Thus, finally, combining (10), (13) and (14), we get

$$\lim_{x \to 1-} \sum_{n=1}^{\infty} \frac{v_n}{n(n+1)} x^{n+1} = A - a_0.$$

But here we can apply the already proved special case of the theorem, obtaining

$$\sum_{n=1}^{\infty} \frac{v_n}{n(n+1)} = A - a_0.$$

On the other hand, we have

$$\sum_{n=1}^{m} \frac{v_n}{n(n+1)} = \sum_{n=1}^{m} \frac{v_n}{n} - \sum_{n=1}^{m} \frac{v_n}{n+1} = \sum_{n=1}^{m} \frac{v_n}{n} - \sum_{n=1}^{m+1} \frac{v_{n-1}}{n}$$

$$= -\frac{v_m}{m+1} + \sum_{n=1}^{m} a_n.$$

Since the first term on the right approaches zero as $m \to \infty$, it follows that

$$\lim_{m \to \infty} \sum_{n=1}^{m} a_n = A - a_0. \quad \blacksquare$$

Remark. Subsequently, various authors have proved a whole series of delicate theorems of the same type (customarily called "Tauberian theorems"), modifying and extending Tauber's conditions.

PROBLEMS

1. Show that the series

$$\frac{1}{2} + \sum_{n=1}^{\infty} \cos n\theta$$

diverges for all θ in the interval $[-\pi, \pi]$.

Hint. If θ/π is a rational number, i.e., of the form p/q where p and $q > 0$ are integers, then $\cos n\theta = \pm 1$ for values of n which are multiples of q. This violates the necessary condition for convergence of a series (*Ruds.*, Theorem 5, p. 8). If the ratio θ/π is irrational, then, expanding θ/π as an infinite continued fraction, we find that[4]

$$\left| \frac{\theta}{\pi} - \frac{m}{n} \right| < \frac{1}{n^2}$$

for appropriate rational numbers m/n, and hence

$$|n\theta - m\pi| < \frac{\pi}{n}.$$

Therefore, for infinitely many values of n, we have

$$|\cos n\theta \pm 1| < \frac{\pi}{n},$$

so that

$$|\cos n\theta| > 1 - \frac{\pi}{n},$$

which again violates the necessary condition for convergence.

4. See A. Y. Khinchin, *Continued Fractions*, third edition, University of Chicago Press (1964), p. 17.

2. Find the generalized sum $A(\theta)$ in the sense of Poisson of the series

$$\sum_{n=1}^{\infty} \sin n\theta \qquad (-\pi \leqslant \theta \leqslant \pi). \tag{15}$$

Hint.

$$\sum_{n=1}^{\infty} x^n \sin n\theta = \frac{x \sin \theta}{1 - 2x \sin \theta + x^2}.$$

Ans.

$$A(\theta) = \begin{cases} 0 & \text{if} \quad \theta = 0, \\ \dfrac{1}{2} \cot \dfrac{\theta}{2} & \text{if} \quad \theta \neq 0. \end{cases}$$

3. Show that the series (15) diverges for all θ in the interval $[-\pi, \pi]$ except $\theta = 0, \pm\pi$.

17. The Method of Arithmetic Means

Next we consider the *method of arithmetic means.* This method of generalized summation, the idea of which is due to Frobenius in its simplest form, is usually associated with the name of Cesàro who developed the method further.

DEFINITION. *Let*

$$\alpha_0 = A_0, \quad \alpha_1 = \frac{A_0 + A_1}{2}, \ldots, \quad \alpha_n = \frac{A_0 + A_1 + \cdots + A_n}{n}, \ldots$$

be the successive arithmetic means of the partial sums

$$A_n = a_0 + a_1 + \cdots + a_n \qquad (n = 0, 1, 2, \ldots)$$

of a given numerical series

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + \cdots + a_n + \cdots,$$

and suppose that

$$\lim_{n \to \infty} \alpha_n = A.$$

Then the number A is called the ***generalized sum*** **(in the sense of Cesàro)** *of the given series.*

Example 1. In the case of the series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots,$$

we have

$$\alpha_{2k} = \frac{k+1}{2k+1}, \quad \alpha_{2k-1} = \frac{1}{2}.$$

Hence $\alpha_n \to \frac{1}{2}$ as $n \to \infty$, and we get the same sum as by the Poisson–Abel method (Example 1, p. 90).

Example 2. The series

$$\frac{1}{2} + \sum_{n=1}^{\infty} \cos n\theta \qquad (-\pi \leqslant \theta \leqslant \pi) \tag{1}$$

has partial sums

$$A_n = \frac{\sin\left(n + \frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} \qquad (\theta \neq 0)$$

(see *Ruds.*, formula (8), p. 82). The arithmetic means are easily calculated. In fact,

$$(n+1)\,\alpha_n = \frac{1}{2\sin\frac{\theta}{2}} \sum_{m=0}^{n} \sin\left(m + \frac{1}{2}\right)\theta$$

$$= \frac{1}{4\sin^2\frac{\theta}{2}} \sum_{m=0}^{n} [\cos m\theta - \cos(m+1)\,\theta]$$

$$= \frac{1 - \cos(n+1)\,\theta}{4\sin^2\frac{\theta}{2}} = \frac{1}{2}\left(\frac{\sin(n+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right)^2,$$

and hence

$$\alpha_n = \frac{1}{2(n+1)} \left(\frac{\sin (n+1)\frac{\theta}{2}}{\sin \frac{\theta}{2}} \right)^2 .$$

Obviously $\alpha_n \to 0$ as $n \to \infty$, so that the generalized sum of the series (1) in the sense of Cesàro equals 0 if $\theta \neq 0$. Thus we get the same result as obtained by the Poisson–Abel method (Example 2, p. 91). This is no coincidence, as will be shown in a moment (see Theorem 1 below).

The fact that the method of summation by arithmetic means *(Cesàro's method)* is *linear* is obvious from the definition. Moreover, if the limit

$$\lim_{n \to \infty} A_n$$

exists and equals A, then

$$\lim_{n \to \infty} \alpha_n = A$$

as well, by Corollary 2, p. 23. It follows that Cesàro's method is also *regular*.

LEMMA. *If the series*

$$\sum_{n=0}^{\infty} a_n$$

is summable by the method of arithmetic means to a finite "sum" A, then

$$a_n = o(n).$$

Proof. Since

$$\alpha_{n-1} \to A, \quad \frac{n+1}{n} \alpha_n \to A$$

as $n \to \infty$, we have

$$\frac{(n+1)\alpha_n - n\alpha_{n-1}}{n} = \frac{A_n}{n} \to 0$$

and hence

$$\frac{a_n}{n} = \frac{A_n}{n} - \frac{n-1}{n}\frac{A_{n-1}}{n-1} \to 0$$

as $n \to \infty$. ∎

We can now give a definitive answer to the question of the connection between the Poisson–Abel method of summation and Cesàro's method:

THEOREM 1 **(Frobenius).** *If the series*

$$\sum_{n=0}^{\infty} a_n$$

is summable by the method of arithmetic means to a finite "sum" A, then it is also summable by the Poisson–Abel method to the same sum A.

Proof. By hypothesis, $\alpha_n \to A$ as $n \to \infty$. Then the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

clearly converges for $0 < x < 1$, by the lemma. Applying Abel's transformation *twice* (as in Prob. 3, p. 17 or *Ruds.*, Prob. 1, p. 85), we get

$$f(x) = (1-x)\sum_{n=0}^{\infty} A_n x^n = (1-x)^2 \sum_{n=0}^{\infty} (n+1)\,\alpha_n x^n. \qquad (2)$$

But, as is easily verified,

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)\,x^n$$

or

$$1 = (1-x)^2 \sum_{n=0}^{\infty} (n+1)\,x^n \qquad (3)$$

for $0 < x < 1$. Multiplying both sides of (3) by A and then subtracting (2) term by term from the resulting identity, we get

$$A - f(x) = (1-x)^2 \sum_{n=0}^{\infty} (n+1)\,(A - \alpha_n)\,x^n.$$

The right-hand side is the sum of the two expressions

$$(1-x)^2\sum_{n=0}^{N}(n+1)(A-\alpha_n)x^n, \quad (1-x)^2\sum_{n=N+1}^{\infty}(n+1)(A-\alpha_n)x^n. \tag{4}$$

Let N be so large that $|A-\alpha_n| < \varepsilon$ for all $n > N$, where ε is any preassigned positive number. Then the second of the expressions (4) is itself less than ε is absolute value (independently of x), while the first expression can also be made less than ε is absolute value by simply choosing x close enough to 1.[5]

Thus we have shown that whenever Cesàro's method is applicable, the Poisson–Abel method is also applicable and leads to the same generalized sum. The converse is not true, i.e., there exist series summable by the Poisson–Abel method but not by Cesàro's method. For example, the series

$$1-2+3-4+\cdots \tag{5}$$

cannot be summed by Cesàro's method, since the necessary condition $a_n = o(n)$ proved in the lemma obviously fails. On the other hand, the power series

$$1-2x+3x^2-4x^3+\cdots \qquad (0<x<1)$$

has the sum

$$\frac{1}{(1+x)^2}$$

which approaches $\frac{1}{4}$ as $x \to 1-$, so that (5) is summable to $\frac{1}{4}$ by the Poisson–Abel method. Thus the Poisson–Abel method is more powerful (i.e., applicable to a larger class of series) than Cesàro's method, but leads to the same result as Cesàro's method whenever both methods are applicable.

Just as in the case of the Poisson–Abel method, we can prove theorems of a "Tauberian" type for Cesàro's method, establish-

5. Note the resemblance between the proof of this theorem and Theorem 1, p. 92.

ing extra conditions on the terms of a series under which the summation of the series by Cesàro's method implies its convergence in the ordinary sense. Because of Frobenius' theorem, it is clear that every Tauberian theorem for the Poisson–Abel method leads to an analogous theorem for Cesàro's method. For example, Tauber's theorem itself (Theorem 2, p. 94) can be paraphrased as follows: *If* $\alpha_n \to A$ *and*

$$\lim_{n\to\infty} \frac{a_1 + 2a_2 + \cdots + na_n}{n} = 0, \tag{6}$$

then $A_n \to A$. Here, however, the theorem is an immediate consequence of the easily verified identity[6]

$$A_n - \alpha_n = \frac{a_1 + 2a_2 + \cdots + na_n}{n+1},$$

which in the present case even implies the *necessity* of (6).

It was proved by Hardy that $\alpha_n \to A$ implies $A_n \to A$ not only if $a_n = o(1/n)$ (this case is contained in the foregoing) but also under the weaker assumption that

$$|ma_m| < C \qquad (m = 1, 2, \ldots) \tag{7}$$

for some constant $C > 0$. Then Landau showed that (7) can even be replaced by the corresponding "one-sided" inequality:

THEOREM 2 **(Hardy–Landau).** *If the series*

$$\sum_{n=0}^{\infty} a_n$$

is summable to A by the method of arithmetic means and if

$$ma_m > -C \qquad (m = 1, 2, \ldots)$$

6. Note that

$$\begin{aligned}(n+1)A_n - (n+1)\alpha_n &= (n+1)A_n - (A_0 + A_1 + \cdots + A_n)\\ &= (A_n - A_0) + (A_n - A_1) + \cdots + (A_n - A_{n-1})\\ &= (a_1 + a_2 + \cdots + a_n) + (a_2 + \cdots + a_n)\\ &\quad + \cdots + a_n\\ &= a_1 + 2a_2 + \cdots + na_n.\end{aligned}$$

for some constant $C > 0$, *then*[7]

$$\sum_{n=0}^{\infty} a_n = A.$$

Proof. First consider the sum

$$S = \sum_{m=n+1}^{n+k} A_m,$$

where n and k are arbitrary positive integers. This sum is easily transformed into

$$S = \sum_{m=0}^{n+k} A_m - \sum_{m=0}^{n} A_m = (n + k + 1)\,\alpha_{n+k} - (n + 1)\,\alpha_n$$

$$= k\alpha_{n+k} + (n + 1)\,(\alpha_{n+k} - \alpha_n). \tag{8}$$

It follows from the hypothesis

$$a_m > -\frac{C}{m}$$

that any A_m (with $n < m \leqslant n + k$) has the following lower bound:

$$A_m = A_n + (a_{n+1} + \cdots + a_n) > A_n - \frac{k}{n} C.$$

Then, summing over m, we find that

$$S > kA_n - \frac{k^2}{n} C.$$

7. Changing the sign of all the terms of the series, we see that it is also sufficient to assume the opposite inequality

$$ma_m < C \qquad (m = 1, 2, \ldots).$$

In particular, the theorem is obviously applicable to all series with terms of *constant* sign.

Comparing this with (8), we find that

$$A_m < \alpha_{n+k} + \frac{n+1}{k}(\alpha_{n+k} - \alpha_n) + \frac{k}{n} C. \tag{9}$$

Now let n approach infinity while at the same time k is subject to the requirement that the ratio k/n approach a preassigned positive number ε. Then the right-hand side of (9) approaches the limit $A + \varepsilon C$, so that

$$A_n < A + 2\varepsilon C \tag{10}$$

for all sufficiently large n.

In just the same way, by considering the sum

$$S' = \sum_{m=n-k+1}^{n} A_m = k\alpha_{n-k} + (n+1)(\alpha_n - \alpha_{n-k})$$

and deducing the upper bound

$$A_m = A_n - (a_{m+1} + \cdots + a_n) < A_n + \frac{k}{n-k} C$$

for A_m, we arrive at the inequality

$$S' < kA_n + \frac{k^2}{n-k} C.$$

It follows that

$$A_n > \alpha_{n-k} + \frac{n+1}{k}(\alpha_n - \alpha_{n-k}) - \frac{k}{n-k} C. \tag{11}$$

If $n \to \infty$ and at the same time $k/n \to \varepsilon$, then as before (but this time let $\varepsilon < \frac{1}{2}$), the right-hand side of (11) approaches the limit

$$A - \frac{\varepsilon}{1-\varepsilon} C > A - 2\varepsilon C,$$

so that

$$A_n > A - 2\varepsilon C \tag{12}$$

for all sufficiently large n. Comparing (10) and (12), we finally get

$$\lim_{n\to\infty} A_n = A. \quad \blacksquare$$

Remark. A similar Tauberian theorem has since been proved for Poisson–Abel summation, with the Hardy–Landau theorem as a special case. Because of the complexity of its proof, this more general theorem will not be given here.

PROBLEMS

1. Use the method of arithmetic means to sum the series

$$\sum_{n=1}^{\infty} \sin n\theta \qquad (-\pi \leqslant \theta \leqslant \pi).$$

Compare the result with Problem 2, p. 98.

Hint. Since

$$A_n = \frac{\cos\frac{\theta}{2} - \cos\left(n + \frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} \qquad (\theta \neq 0).$$

(see *Ruds.*, formula (7), p. 8), we have

$$(n+1)\,\alpha_n = \frac{n+1}{2}\cot\frac{\theta}{2} - \frac{1}{4\sin^2\frac{\theta}{2}} \times$$

$$\times \sum_{m=1}^{n+1} [\sin(m+1)\,\theta - \sin m\theta]$$

$$= \frac{n+1}{2}\cot\frac{\theta}{2} - \frac{\sin(n+2)\,\theta - \sin\theta}{4\sin^2\frac{\theta}{2}}.$$

It follows that

$$\lim_{n\to\infty} \alpha_n = \frac{1}{2}\cot\frac{\theta}{2} \qquad (\theta \neq 0).$$

2. Prove formula (2) directly.

Ans.

$$(1 - 2x + x^2)\sum_{n=0}^{\infty}(n+1)\alpha_n x^n$$

$$= \sum_{n=0}^{\infty}[(n+1)\alpha_n - 2n\alpha_{n-1} + (n-1)\alpha_{n-2}]\,x^n$$

$$= \sum_{n=0}^{\infty}\{[(n+1)\alpha_n - n\alpha_{n-1}] - [n\alpha_{n-1} - (n-1)\alpha_{n-2}]\}\,x^n$$

$$= \sum_{n=0}^{\infty}(A_n - A_{n-1})\,x^n = \sum_{n=0}^{\infty} a_n x^n,$$

where the series on the left converges by the boundedness of the α_n and we set $\alpha_{-1} = \alpha_{-2} = A_{-1} = 0$ (the convergence of the last series is automatic here).

3. Let $\{a_n\}$ be a positive sequence decreasing monotonically to zero, and let

$$A_0 = a_0, \qquad A_n = a_0 + a_1 + \cdots + a_n \qquad (n > 0).$$

Prove that the alternating series

$$A_0 - A_1 + A_2 - A_3 + \cdots \tag{13}$$

is summable by Cesàro's method and that its generalized sum equals one half the sum α of the convergent series

$$a_0 - a_1 + a_2 - a_3 + \cdots$$

of the Leibniz type.

Hint. The arithmetic mean of the first $2m$ partial sums of (13) can be written in the form

$$\frac{1}{2}\,\frac{(a_0-a_1)+\cdots+(a_0-a_1+\cdots+a_{2m-2}-a_{2m-1})}{m},$$

and hence approaches $\frac{1}{2}\alpha$ by Corollary 2, p. 23. Show that the arithmetic mean of the first $2m+1$ partial sums of (13) approaches the same limit.

4. Show that the divergent series

$$H_1 - H_2 + H_3 - H_4 + \cdots,$$

where

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n},$$

is summable by Cesàro's method to $\frac{1}{2}\ln 2$.

Hint. Use the preceding problem with

$$a_n = \frac{1}{n+1} \quad (n = 0, 1, 2, \ldots).$$

5. Show that the divergent series

$$\ln 2 - \ln 3 + \ln 4 - \ln 5 +$$

is summable by Cesàro's method to $\frac{1}{2}\ln(\pi/2)$.

Hint. Use Problem 3 with

$$a_n = \ln\frac{n+2}{n+1} \qquad (n = 0, 1, 2, \ldots),$$

recalling Wallis' product (*Ruds.*, Prob. 1, p. 91).

6. Use Problem 3 to prove that the divergent Dirichlet series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^x} \equiv \sum_{n=1}^{\infty}(-1)^{n-1}\,n^{\xi}$$

$$(-1 < x < 0, \quad \xi = -x, \; 0 < \xi < 1)$$

(cf. *Ruds.*, Example 2, p. 82) is summable by Cesàro's method.

Hint. Writing n^ξ as the sum

$$n^\xi = (1 - 0) + (2^\xi - 1) + \cdots + (n^\xi - (n - 1)^\xi),$$

use differentiation to show that $f(n) = n^\xi - (n - 1)^\xi$ is a decreasing function of n. Then show that $f(n) \to 0$ as $n \to \infty$.

7. Consider the series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots, \tag{14}$$

$$1 - 1 + 0 + 1 - 1 + 0 + 1 - 1 + 0 + \cdots, \tag{15}$$

$$0 + 1 - 1 + 0 + 1 + 0 + 0 + 0 - 1 + \cdots, \tag{16}$$

where (15) differs from (14) by having a zero after every pair of terms $+1 - 1$, while (16) differs from (14) in that the term ± 1 in the mth position ($m = 0, 1, 2, \ldots$) is moved to the 2^mth position with the remaining positions being filled by zeros. As we already know, (14) is summable by Cesàro's method to $\frac{1}{2}$. Show that (15) is summable by Cesàro's method to $\frac{1}{3}$, while (16) fails to be summable by Cesàro's method.

Hint. In the case of the series (16), as n goes from 2^{2m-1} to $2^{2m} - 1$, the arithmetic mean of the first $n + 1$ partial sums oscillates from

$$\frac{1}{3}\,\frac{2^{2m} - 1}{2^{2m+1} + 1} \to \frac{2}{3}$$

to

$$\frac{1}{3}\,\frac{2^{2m} - 1}{2^{2m}} \to \frac{1}{3}.$$

Comment. Separating the terms of a convergent series by clusters of zeros has no effect either on the convergence of the series or on its sum. This problem shows that the situation is quite different in the case of summation of divergent series.

18. Application of Generalized Summation to Multiplication of Series

We now apply the method of generalized summation to the problem of *multiplication of series* (Sec. 4). Thus, given two series

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + \cdots + a_n + \cdots, \tag{1}$$

$$\sum_{n=0}^{\infty} b_n = b_0 + b_1 + \cdots + b_n + \cdots, \tag{2}$$

we form the series

$$\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0), \tag{3}$$

.e., the product of (1) and (2) in Cauchy's form (see p. 15). Even if (1) and (2) converge and have ordinary sums A and B, the series (3) may fail to converge (see Example 1, p. 27). However, we still have the following

THEOREM. *If the series* (1) *and* (2) *converge and have ordinary sums A and B, then the series* (3) *is summable by the Poisson–Abel method to the generalized sum AB.*

Proof. The series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \cdots + a_n x^n + \cdots \tag{1'}$$

and

$$\sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + \cdots + b_n x^n + \cdots \tag{2'}$$

both converge absolutely for $0 < x < 1$ (*Ruds.*, Lemma, p. 68) to sums $f(x)$ and $g(x)$, respectively. Hence the product of (1') and (2'), i.e., the series

$$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0)\, x^n$$

also converges for $0 < x < 1$ and has the sum $f(x)\,g(x)$, by the classical Cauchy theorem (p. 14). But

$$\lim_{x \to 1-} f(x) = A, \qquad \lim_{x \to 1-} g(x) = B,$$

by Theorem 1, p. 92, and hence

$$\lim_{x \to 1-} f(x)\,g(x) = AB,$$

i.e., the generalized sum (in the Poisson–Abel sense) of the series (3) actually exists and equals AB. ∎

The theorem just proved implies Abel's theorem on the multiplication of series (Theorem 2, p. 29). Moreover, it is clear from the proof that the assumption that (1) and (2) converge and have ordinary sums A and B can be replaced by the assumption that (1) and (2) are summable by the Poisson–Abel method to the generalized sums A and B. Recalling Frobenius' theorem (Theorem 1, p. 101), we conclude that *if the series* (1), (2) *and* (3) *are summable by Cesàro's method to generalized sums* A, B *and* C, *respectively, then*

$$C = AB.$$

Example 1. Consider the series

$$\frac{1}{\sqrt{2}} = 1 - \frac{1}{2} + \frac{3}{8} - \cdots + (-1)^n \frac{(2n-1)(2n-3)\cdots 3 \cdot 1}{2n\,(2n-2)\cdots 4 \cdot 2} + \cdots, \tag{4}$$

obtained by setting $x = 1$ in the binomial expansion

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \cdots + (-1)^n \frac{(2n-1)(2n-3)\cdots 3 \cdot 1}{2n\,(2n-2)\cdots 4 \cdot 2}\,x^n + \cdots$$

(*Ruds.*, Prob. 3c, p. 123). Multiplying (4) by itself (see Problem 1), we get the familiar series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots, \tag{5}$$

whose generalized sum both in the Poisson–Abel sense and in the Cesàro sense equals

$$\frac{1}{2} = \left(\frac{1}{\sqrt{2}}\right)^2.$$

Example 2. "Squaring" the divergent series (5), we get the series

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots, \tag{6}$$

which is Poisson–Abel summable to the sum

$$\frac{1}{4} = \left(\frac{1}{2}\right)^2,$$

but not Cesàro summable!

PROBLEMS

1. Prove that

$$\sum_{m=0}^{n} \frac{(2m-1)(2m-3)\cdots 3\cdot 1}{2m(2m-2)\cdots 4\cdot 2} \times$$

$$\times \frac{(2n-2m-1)(2n-2m-3)\cdots 3\cdot 1}{(2n-2m)(2n-2m-2)\cdots 4\cdot 2} = 1,$$

where $(2m-1)(2m-3)\cdots 3\cdot 1$ and $2m(2m-2)\cdots 4\cdot 2$ are formally equal to 1 if $m = 0$.

2. What is the generalized sum in the Poisson–Abel sense of the product of the divergent series (5) and (6)? Write this product.

19. Other Methods of Generalized Summation

We now consider a number of other methods of generalized summation, as applied to a given numerical series

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + \cdots + a_n + \cdots \tag{1}$$

with partial sums

$$A_n = a_0 + a_1 + \cdots + a_n.$$

1) *Voronoi's method.* Given a sequence of positive numbers $\{q_n\}$, let

$$Q_0 = q_0, \quad Q_n = q_0 + q_1 + \cdots + q_n \qquad (n > 0),$$

and form the expressions

$$w_n = \frac{q_n A_0 + q_{n-1} A_1 + \cdots + q_0 A_n}{Q_n} \qquad (n = 0, 1, 2, \ldots).$$

If $w_n \to A$ as $n \to \infty$, then A is called the "generalized sum" o the series (1) in the sense of Voronoi, for the given choice of the sequence $\{q_n\}$. The linearity of Voronoi's method (and of all the other methods considered below) is obvious, and need not be discussed further. As for the regularity of the method, we have

THEOREM 1. *A necessary and sufficient condition for regularity of Voronoi's method is that*

$$\lim_{n\to\infty} \frac{q_n}{Q_n} = 0. \tag{2}$$

Proof. First we prove the necessity, assuming that the method is regular, i.e., that $A_n \to A$ always implies $w_n \to A$. In particular, choosing the series

$$1 - 1 + 0 + 0 + 0 + \cdots,$$

for which $A_0 = 0$ and all other $A_n = 0$ (so that $A = 0$), we get

$$w_n = \frac{q_n}{Q_n} \to 0$$

as $n \to \infty$, as required.

To prove the sufficiency, suppose (2) holds. Let

$$t_{nn} = \frac{q_{n-m}}{Q_n}.$$

Then $t_{nm} \to 0$ as $n \to \infty$ (m fixed), since

$$t_{n-m} = \frac{q_{n-m}}{Q_n} < \frac{q_{n-m}}{Q_{n-m}} \to 0$$

as $n \to \infty$, while

$$\sum_{m=0}^{n} t_{nm} = \sum_{m=0}^{n} |t_{nm}| = 1.$$

It follows from Corollary 1, p. 23 that

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} (t_{n0}A_0 + t_{n1}A_1 + \cdots + t_{nn}A_n) = \lim_{n\to\infty} A_n = A. \quad \blacksquare$$

2) *The generalized Cesàro method.* The method of arithmetic means encountered in Sec. 17 is the simplest of an infinite family of summation methods due to Cesàro. Given a fixed positive integer k, let

$$\gamma_n^{(k)} = \frac{S_n^{(k)}}{C_k^{n+k}} = \frac{C_{k-1}^{n+k-1}A_0 + C_{k-1}^{n+k-2}A_1 + \cdots + C_{k-1}^{k-1}A_n}{C_k^{n+k}},$$

where as usual C_m^n is the binomial coefficient

$$C_m^n = \frac{n!}{m!\,(n-m)!}.$$

Then the limit of $\gamma_n^{(k)}$ as $n \to \infty$ (provided it exists) is regarded as the "generalized sum" (of order k) of the series (1). For $k = 1$, this reduces to the method of arithmetic means.

The generalized Cesàro method is a special kind of regular Voronoi method. To see this, we need only set

$$q_n = C_{k-1}^{n+k-1},$$

since it follows from the formula

$$C_k^{n+k} = C_{k-1}^{k-1} + C_{k-1}^{k} + C_{k-1}^{k+1} + \cdots + C_{k-1}^{n+k-1} \tag{3}$$

(see Problem 1) that

$$Q_n = C_k^{n+k},$$

so that

$$\frac{q_n}{Q_n} = \frac{C_{k-1}^{n+k-1}}{C_k^{n+k}} = \frac{k}{n+k} \to 0$$

as $n \to \infty$.

Using the definition of $S_n^{(k)}$ and the identity (3), we easily find that

$$S_0^{(k-1)} + S_1^{(k-1)} + \cdots + S_n^{(k-1)} = S_n^{(k)}, \tag{4}$$

where $S_n^{(0)} = A_n$. This allows us to prove

THEOREM 2. *Suppose the series*

$$\sum_{n=0}^{\infty} a_n$$

is summable to A by the generalized Cesàro method of order $k-1$. *Then the series is also summable to A by the generalized Cesàro method of order k (and hence of any order* $\geqslant k$*).*

Proof. By hypothesis, $\gamma_n^{(k-1)} \to A$ as $n \to \infty$. But

$$\gamma_n^{(k)} = \frac{S_n^{(k)}}{C_k^{n+k}} = \frac{S_0^{(k-1)} + S_1^{(k-1)} + \cdots + S_n^{(k-1)}}{C_k^{n+k}}$$

$$= \frac{C_{k-1}^{k-1}\gamma_0^{(k-1)} + C_{k-1}^{k}\gamma_1^{(k-1)} + \cdots + C_{k-1}^{n+k-1}\gamma_n^{(k-1)}}{C_k^{n+k}},$$

because of (4). Using (3) to apply Corollary 1, p. 23, with

$$x_n = \gamma_n^{(k-1)}, \quad t_{nm} = \frac{C_{k-1}^{m+k-1}}{C_k^{n+k}} \quad (m = 0, 1, \ldots, n),$$

we find that $\gamma_n^{(k)} \to A$ as $n \to \infty$. ∎

Next we prove the following natural generalization of Frobenius' theorem (Theorem 1, p. 101):

THEOREM 3. *If the series*

$$\sum_{n=0}^{\infty} a_n$$

is summable by the generalized Cesàro method of any order k to a finite "sum" A, then it is also summable by the Poisson–Abel method to the same sum A.

Proof. By hypothesis,

$$\lim_{n\to\infty} \gamma_n^{(k)} = \lim_{n\to\infty} \frac{S_n^{(k)}}{C_k^{n+k}} = A. \tag{5}$$

It is easy to see that the power series

$$\sum_{n=0}^{\infty} S_n^{(k)} x^n \tag{6}$$

converges for $-1 < x < 1$. In fact, since

$$C_k^{n+k} \sim \frac{n^k}{k!}$$

as $n \to \infty$, it follows from (5) that

$$\lim_{n\to\infty} \frac{|S_n^{(k)}|}{n^k} = \frac{|A|}{k!}.$$

If $A \neq 0$, then

$$\lim_{n\to\infty} \sqrt[n]{|S_n^{(k)}|} = 1,$$

so that the radius of convergence of the series (6) equals 1, by the Cauchy–Hadamard theorem (*Ruds.*, p. 70), while if $A = 0$, the radius of convergence of (6) is infinite and hence obviously no less than 1.

Now consider the series of identities

$$\sum_{n=0}^{\infty} a_n x^n = (1-x)\sum_{n=0}^{\infty} A_n x^n = (1-x)\sum_{n=0}^{\infty} S_n^{(0)} x^n,$$

$$\sum_{n=0}^{\infty} S_n^{(0)} x^n = (1-x)\sum_{n=0}^{\infty} S_n^{(1)} x^n, \tag{7}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum_{n=0}^{\infty} S_n^{(k-1)} x^n = (1-x)\sum_{n=0}^{\infty} S_n^{(k)} x^n,$$

where we repeatedly make use of formula (4), p. 92 and formula (4) above. Since the series (6) converges in the interval $(-1, 1)$, so do all the other series in (7).[8] Combining the identities (6), we get

$$\sum_{n=0}^{\infty} a_n x^n = (1-x)^{k+1}\sum_{n=0}^{\infty} S_n^{(k)} x^n = (1-x)^{k+1}\sum_{n=0}^{\infty} C_k^{n+k}\gamma_n^{(k)} x^n. \tag{8}$$

Moreover, by multiplying the series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

by itself $k+1$ times, we get the identity

$$1 = (1-x)^{k+1}\sum_{n=0}^{\infty} C_n^{n+k} x^n, \tag{9}$$

which holds in the same interval $(-1, 1)$. Multiplying both sides of (9) by A and then subtracting (8) term by term from the resulting identity, we finally obtain

$$A - \sum_{n=0}^{\infty} a_n x^n = (1-x)^{k+1}\sum_{n=0}^{\infty} C_k^{n+k}(A - \gamma_n^{(k)})\, x^n.$$

8. Cf. the comment to Problem 3, p. 17.

The rest of the proof, with the use of (5), is the exact analogue of the proofs of Abel's theorem (Theorem 1, p. 92) and Frobenius' theorem (Theorem 1, p. 101). As a result, we conclude that

$$\lim_{x \to 1-} \sum_{n=0}^{\infty} a_n x^n = A. \quad \blacksquare$$

Remark. There exist divergent series which are summable by the Poisson–Abel method, but not by the generalized Cesàro method of any order. Thus the Poisson–Abel method is more powerful than the generalized Cesàro method of arbitrarily high order.

PROBLEMS

1. Prove formula (3).

Hint. Use mathematical induction on n, starting from the easily verified formula

$$C_k^{n+k} = C_k^{(n-1)+k} + C_{k-1}^{n+(k-1)}.$$

2. Given any positive integer k, show that the series

$$\sum_{n=0}^{\infty} (-1)^n C_k^{n+k} \tag{10}$$

is not summable by the generalized Cesàro method of order k, but is summable to $1/2^{k+1}$ by the generalized Cesàro method of order $k + 1$.

Hint. It follows from (8) and (9) that

$$\sum_{n=0}^{\infty} S_n^{(k)} x^n = \frac{1}{(1-x)^{k+1}} \sum_{n=0}^{\infty} a_n x^n = \frac{1}{(1-x)^{k+1}} \sum_{n=0}^{\infty} (-1)^n C_k^{n+k} x^n$$

$$= \frac{1}{(1-x^2)^{k+1}} = \sum_{m=0}^{\infty} C_k^{m+k} x^{2m},$$

where the convergence of the last series in the interval $(-1, 1)$ is easily proved by using the Cauchy–Hadamard theorem (this

implies the convergence of the first series as well). Equating coefficients of identical powers of x in the first and last series,[9] we get

$$S_{2m}^{(k)} = C_k^{m+k}, \quad S_{2m+1}^{(k)} = 0 \qquad (m = 0, 1, 2, \ldots). \tag{11}$$

Therefore

$$\lim_{m\to\infty} \gamma_{2m}^{(k)} = \lim_{m\to\infty} \frac{C_k^{m+k}}{C_k^{2m+k}} = \frac{1}{2^k},$$

while

$$\lim_{m\to\infty} \gamma_{2m+1}^{(k)} = 0,$$

so that (10) fails to be summable by the generalized Cesàro method of order k. On the other hand, it follows from (3), (4) and (11) that

$$S_n^{(k+1)} = C_k^k + C_k^{k+1} + C_k^{k+2} + \cdots + C_k^{m+k} = C_{k+1}^{m+k+1}$$

both for $n = 2m$ and $n = 2m + 1$. Hence

$$\lim_{m\to\infty} \gamma_{2m}^{(k+1)} = \lim_{m\to\infty} \frac{C_{k+1}^{m+k+1}}{C_{k+1}^{2m+k+1}} = \frac{1}{2^{k+1}},$$

and similarly for $\gamma_{2m+1}^{(k+1)}$.

3. Prove that the series

$$\sum_{n=0}^{\infty} (-1)^n (n+1)^k,$$

where k is any positive integer, is summable by the generalized Cesàro method of order $k + 1$.

Hint. Expanding C_k^{n+k} in powers of $n + 1$, we get the formula

$$C_k^{n+k} = \frac{1}{k!}(n+k)(n+k-1)\cdots(n+1)$$

$$= \alpha_1^{(k)}(n+1)^k + \alpha_2^{(k)}(n+1)^{k-1} + \cdots + \alpha_{k-1}^{(k)}(n+1),$$

9. Here we anticipate the legitimacy of this operation, to be proved in G. M. Fichtenholz, *Functional Series*, Sec. 6, Theorem 3.

involving certain numerical coefficients $\alpha_l^{(k)}$, where

$$\alpha_1^{(k)} = \frac{1}{k!} \neq 0.$$

Writing similar formulas with k replaced by $k-1, k-2, \ldots, 1$, and then solving for $(n+1)^k$, we find that

$$(n+1)^k = \beta_1^{(k)} C_k^{n+k} + \beta_2^{(k)} C_{k-1}^{n+k-1} + \cdots + \beta_k^{(k)} C_1^{n+1},$$

where the $\beta_1^{(k)}$ are again certain numerical coefficients. But then

$$\sum_{n=0}^{\infty} (-1)^n (n+1)^k = \beta_1^{(k)} \sum_{n=0}^{\infty} (-1)^n C_k^{n+k} + \beta_2^{(k)} \sum_{n=0}^{\infty} (-1)^n C_{k-1}^{n+k-1}$$

$$+ \cdots + \beta_k^{(k)} \sum_{n=0}^{\infty} (-1)^n C_1^{n+1}.$$

Now use the preceding problem and Theorem 2, as well as the linearity of the given method of summation.

4. By *Hölder's method* of generalized summation is meant repeated application of the method of arithmetic means. Thus, besides

$$\alpha_0 = A_0, \quad \alpha_1 = \frac{A_0 + A_1}{2}, \ldots, \quad \alpha_n = \frac{A_0 + A_1 + \cdots + A_n}{n+1}, \ldots,$$

as on p. 98, we have next

$$\alpha_0' = \alpha_0, \quad \alpha_1' = \frac{\alpha_0 + \alpha_1}{2}, \ldots, \quad \alpha_n' = \frac{\alpha_0 + \alpha_1 + \cdots + \alpha_n}{n+1}, \ldots$$

with corresponding generalized sum

$$\lim_{n \to \infty} \alpha_n',$$

then

$$\alpha_0'' = \alpha_0', \quad \alpha_1'' = \frac{\alpha_0' + \alpha_1'}{2}, \ldots, \quad \alpha_n'' = \frac{\alpha_0' + \alpha_1' + \cdots + \alpha_n'}{n+1}, \ldots$$

with corresponding generalized sum

$$\lim_{n\to\infty} \alpha_n'',$$

and so on. Sum the following series by Hölder's method:

a) $1 - 2 + 3 - 4 + \cdots$; b) $1 - 3 + 6 - 10 + \cdots$.

Ans. a) Twofold "averaging" gives $\frac{1}{4}$; b) threefold "averaging" gives $\frac{1}{8}$.

Comment. The regularity of Hölder's method and the relation between the various generalized sums can be found by using Corollary 2, p. 23. It can be shown that k-fold application of the method of arithmetic means is completely equivalent to the generalized Cesàro method of order k, i.e., every series summable by one method is summable to the same sum by the other method.

20. The Methods of Borel and Euler

1) *Borel's method.* This method consists in forming the expression

$$\frac{\sum_{n=0}^{\infty} A_n \frac{x^n}{n!}}{\sum_{n=0}^{\infty} \frac{x^n}{n!}} = e^{-x} \sum_{n=0}^{\infty} A_n \frac{x^n}{n!}, \tag{1}$$

involving the partial sums

$$A_n = a_0 + a_1 + \cdots + a_n$$

of the given numerical series

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + \cdots + a_n + \cdots. \tag{2}$$

Suppose the series in the right-hand side of (1) converges, at least for sufficiently large x, and suppose the expression

$$e^{-x} \sum_{n=0}^{\infty} A_n \frac{x^n}{n!}$$

approaches a finite limit A as $x \to +\infty$. Then the number A is called the *generalized sum (in the sense of Borel)* of the given series (2).

THEOREM. *Borel's method of summation is regular.*

Proof. Suppose the series (2) is convergent, with sum A. Then, for sufficiently large x,

$$A - e^{-x} \sum_{n=0}^{\infty} A_n \frac{x^n}{n!} = e^{-x} \sum_{n=0}^{\infty} A \frac{x^n}{n!} - e^{-x} \sum_{n=0}^{\infty} A_n \frac{x^n}{n!}$$

$$= e^{-x} \sum_{n=0}^{\infty} \alpha_n \frac{x^n}{n!},$$

where $\alpha_n = A - A_n$. The right-hand side is the sum of the two expressions

$$e^{-x} \sum_{n=0}^{N} \alpha_n \frac{x^n}{n!}, \quad e^{-x} \sum_{n=N+1}^{\infty} \alpha_n \frac{x^n}{n!} \tag{3}$$

(here we use the same technique as in the proof of Abel's theorem, p. 92 and Frobenius' theorem, p. 101). Given any $\varepsilon > 0$, let N be so large that $|\alpha_n| < \varepsilon$ for all $n > N$. Then the second of the expressions (3) is less than ε in absolute value (independently of x), while the first expression, being the product of e^{-x} and a polynomial, can be made less than ε in absolute value by simply choosing x large enough. ∎

2) *Euler's method.* It will be recalled from formula (7), p. 74 that

$$\sum_{k=0}^{\infty} (-1)^k a_k = \sum_{p=0}^{\infty} (-1)^p \frac{\Delta^p a_0}{2^{p+1}} \tag{4}$$

(Euler's transformation). As shown in Sec. 13, *if the series on the left converges, then the series on the right converges to the same*

sum. However, the series on the right may converge to a finite sum A even if the series on the left diverges. The number A is then called the *generalized sum (in the sense of Euler)* of the series on the left. The regularity of this method of summation, known as *Euler's method*, is an immediate consequence of the italicized assertion.

Starting from the series (4) without the factors of ± 1 and recalling formula (2), p. 71 for the pth differences, we see that the generalized sum in the sense of Euler of the series (2) is just the ordinary sum (provided it exists) of the series

$$\sum_{p=0}^{\infty} \frac{a_0 + C_1^p a_1 + C_2^p a_2 + \cdots + C_p^p a_p}{2^{p+1}}.$$

Remark. This concludes our discussion of methods for summing divergent series. We have presented enough material to give the reader some idea of the great variety of approaches to the problem. Note that we have always insisted that our methods of generalized summation be regular. Had space permitted, more could have been said about the relations between different methods of summation. It may happen that two methods of summation have overlapping "domains of applicability" (neither of which is contained in the other, and it may even turn out that two methods assign different "generalized sums" to the same divergent series.

PROBLEMS

1. Sum the series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

by Borel's method.

Ans.

$$\lim_{x \to \infty} e^{-x} \frac{e^x + e^{-x}}{2} = \frac{1}{2}.$$

2. Sum the following series by Euler's method:

a) $1 - 1 + 1 - 1 + \cdots$; b) $1 - 2 + 3 - 4 + \cdots$;

c) $1 - 2 + 2^2 - 2^3 + \cdots$; d) $1^3 - 2^3 + 3^3 - 4^4 + \cdots$.

Hint. In each case, use formula (4).

Ans. a) $A = \frac{1}{2}$; b) $\Delta^0 a_0 = 1, \Delta^1 a_0 = 1, \Delta^p a_0 = 0$ if $p > 1$, $A = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$; c) $\Delta^p a_0 = 1$, $A = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \cdots = \frac{1}{3}$; d) $\Delta^0 a_0 = 1, \Delta^1 a_0 = 7, \Delta^2 a_0 = 12, \Delta^3 a_0 = 6, \Delta^p a_0 = 0$ if $p > 3$, $A = \frac{1}{2} - \frac{7}{4} + \frac{12}{8} - \frac{6}{16} = -\frac{1}{8}$.

3. Let

$$\varphi_0(x), \varphi_1(x), \ldots, \varphi(x), \ldots$$

be a sequence of functions defined on a domain X, where X has a (finite or positively infinite) limit point ω. Given a numerical series (2), with partial sums $A_n = a_0 + a_1 + \cdots + a_n$, construct the "functional series"

$$\sum_{n=0}^{\infty} A_n \varphi_n(x) = A_0 \varphi_0(x) + A_1 \varphi_1(x) + \cdots + A_n \varphi_n(x) + \cdots. \tag{5}$$

Suppose the series (5) converges, at least for x sufficiently near ω, and suppose the sum of (5) approaches a finite limit A as $x \to \omega$. Finally, let A be called the "generalized sum" of the series (2). This gives a very general method of summation, which is obviously linear. Prove that the method is also regular, provided the functions $\varphi_n(x)$ satisfy the following three conditions:

1) For every fixed n,

$$\lim_{x \to \omega} \varphi_n(x) = 0;$$

2) For x close enough to ω (i.e., for $|x - \omega| < \delta$ if ω is finite or $x > \Delta$ if $\omega = +\infty$),

$$\sum_{n=0}^{\infty} |\varphi_n(x)| \leqslant K = \text{const};$$

3)

$$\lim_{x \to \omega} \sum_{n \to 0}^{\infty} \varphi_n(x) = 1.$$

Hint. If $A_n \to A$ as $n \to \infty$, then, given any $\varepsilon > 0$, there is an N so large that

$$|A_n - A| < \frac{\varepsilon}{K} \tag{6}$$

for all $n > N$. It follows from the boundedness of the A_n and the absolute convergence of the series

$$\sum_{n=0}^{\infty} \varphi_n(x)$$

that the series

$$\sum_{n=0}^{\infty} A_n \varphi_n(x)$$

converges (at least for x close enough to ω). Moreover,

$$\sum_{n=0}^{\infty} A_n \varphi_n(x) - A = \sum_{n=0}^{N} (A_n - A)\varphi_n(x) + \sum_{n=N+1}^{\infty} (A_n - A)\varphi_n(x) + A\left(\sum_{n=0}^{\infty} \varphi_n(x) - 1\right),$$

and hence

$$\left|\sum_{n=0}^{\infty} A_n \varphi_n(x) - A\right| \leqslant \left|\sum_{n=0}^{N} (A_n - A)\varphi_n(x)\right| + \sum_{n=N+1}^{\infty} |A_n - A|\,|\varphi_n(x)| + |A|\left|\sum_{n=0}^{\infty} \varphi_n(x) - 1\right|.$$

The second term on the right is less than ε because of (6), while the first and third terms can each be made less than ε by choosing x close enough to ω.

Comment. Clearly, there is another version of the foregoing which involves the numbers a_n (the terms of the series) rather than the partial sums A_n.

4. Particularize the preceding problem to the case where X is the set of all nonnegative integers m, with limit point $\omega = +\infty$.

Hint. Instead of the sequence of functions $\{\varphi_n(x)\}$, we now have an infinite rectangular matrix

$$\begin{vmatrix} t_{00} & t_{01} & t_{02} & \cdots & t_{0m} & \cdots \\ t_{10} & t_{11} & t_{12} & \cdots & t_{1m} & \cdots \\ t_{20} & t_{21} & t_{22} & \cdots & t_{2m} & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdots \\ t_{n0} & t_{n1} & t_{n2} & \cdots & t_{nm} & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdots \end{vmatrix} \tag{7}$$

($t_{nm} = \varphi_n(m)$), and the "generalized sum" of the series (2) becomes

$$\lim_{m\to\infty} (A_0 t_{0m} + A_1 t_{1m} + A_2 t_{2m} + \cdots + A_n t_{nm} + \cdots).$$

The regularity conditions are now the following:

1) For every fixed n,

$$\lim_{m\to\infty} t_{nm} = 0;$$

2) For sufficiently large m,

$$\sum_{n=0}^{\infty} |t_{nm}| \leqslant K = \text{const};$$

3)

$$\lim_{m\to\infty} \sum_{n=0}^{\infty} t_{nm} = 1.$$

Comment. These ideas are essentially due to Toeplitz, who, however, considered the case where the matrix (7) reduces to an infinite *triangular* matrix. This special case is sufficient for most purposes.

5. Show that the general method of summation of Problem 3 contains the Poisson–Abel method and the Borel method as special cases.

Ans. The Poisson–Abel method corresponds to

$$\varphi_n(x) = (1 - x)\, x^n, \quad X = (0, 1), \quad \omega = 1,$$

while the Borel method corresponds to

$$\varphi_n(x) = e^{-x} \frac{x^n}{n!}, \quad X = (0, +\infty), \quad \omega = +\infty$$

(the regularity conditions are clearly satisfied in both cases).

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